

A generalized triangular form and its global controllability

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Abstract.

We investigate a new class of nonlinear control systems of O.D.E., which are not feedback linearizable in general. Our class is a generalization of the well-known feedback linearizable systems, and moreover it is a generalization of the triangular (or pure-feedback) forms studied before. The definition of our class is global, and coordinate-free, which is why the problem of the equivalence is solved for our class in the whole state space at the very beginning. The goal of this paper is to prove the global controllability of our nonlinear systems. We propose to treat our class as a new canonical form which is a nonlinear global analog of the Brunovsky canonical form on the one hand, and is a global and coordinate-free generalization of the triangular form on the other hand.

Key words: Nonlinear control, triangular form, global controllability, feedback linearization.

Mathematics subject classification: 93C10, 93B10, 93B11, 93B05, 93B52.

1. Introduction.

One of the most important problems in the nonlinear control theory is the problem of classification of nonlinear control systems of O.D.E., i.e., that of finding the transformation of a nonlinear control system into its simplest canonical form along with finding such canonical forms by using some invariants which do not depend on the choice of local coordinates. Beginning with [12],[11], a complete theory of feedback linearization was created – [5],[36], [22], [9], [10],[6], [26], etc. However, even some simple mechanical systems do not satisfy the Respondek-Jakubczyk-Hunt-Su-Meyer conditions; in addition, the concept of feedback linearization is essentially local. This inspired many authors to further investigations and to attempts to generalize the feedback linearization theory.

One possible approach, which is very popular and has various applications, is the concept of differential flatness [8]. However, this notion is as local as that of feedback linearization, and moreover no general criterion of differential flatness has been obtained.

Another way is to deal with the triangular, or pure-feedback form instead of the Brunovsky canonical form. Triangular systems were introduced in [15] as early as 1973 (i.e., even before [12],[11]) as a first example of a nonlinear system which is feedback linearizable. Nevertheless, a triangular system

$$\begin{cases} \dot{z}_i = f_i(z_1, \dots, z_{i+1}), & i = 1, \dots, n-1; \\ \dot{z}_n = f_n(z_1, \dots, z_n, v); \end{cases}$$

is feedback linearizable only in the so-called "regular" case, i.e., when the conditions of regularity $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, $i=1, \dots, n$ hold; otherwise (the "singular case"), the system does not satisfy the Respondek-Jakubczyk-Hunt-Su-Meyer conditions, and, therefore, is not feedback linearizable. The singular case was investigated by Respondek in 1986 (see [31]), and by Celikovsky and Nijmeijer in 1996 (see [4]). In these works, the triangular systems are studied under the assumption that the set of the regular points is open and dense in the whole state space, however. This is not true even for some simple examples (see, for instance system (10) from the current paper).

That is why, we want to find and to investigate a generalization, of the triangular form, which contains all the previous triangular forms studied before (including the singular case) on the one hand, and which can be investigated globally (including the problems of controllability, stabilization, feedback equivalence, etc.) on the other hand. We offered such a generalization in [20], and solved completely the problem of global robust controllability for this class (moreover, the controls constructed were actually closed-loop to some extent). However, the problem of global equivalence of a control system to a system from [20] remained open. In this work, we introduce a generalization of the triangular systems considered in [20], in global coordinate-free terms. The main goal of the current paper is to prove that our generalized triangular form is globally controllable.

In the future, we want to investigate in more detail the relationship between the triangular form from [20] and the class from the current paper. As we can see from example 3.1, the class of "generalized triangular form" is wider than that from [20]. On the other hand, the construction of example 3.1 is based on triangular system (10). To what extent our generalized triangular form can be reduced to the triangular form in the singular case remains an open question.

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2. Notation, and preliminaries.

Let \mathcal{M} be a smooth manifold of dimension n , and $x \mapsto v(x)$ be a smooth vector field on \mathcal{M} . In general, $v(\cdot)$ can be defined on some open subset of \mathcal{M} only; next we denote this subset by \mathcal{D}_v . Let x^0 be in \mathcal{M} . By $I \ni t \mapsto \Phi_v^t(x^0)$ we denote the (maximal) integral curve

$t \mapsto x(t)$ of $\dot{x} = v(x)$ with $x(0) = x^0$. Of course, for each $t \in I$ the map $x \mapsto \Phi_v^t(x)$ is (at least) a diffeomorphism of some neighborhood of x^0 onto some neighborhood of $\Phi_v^t(x^0)$ (and, if, for some $s \in I$, $\Phi_v^s(x)$ is well-defined for all $x \in \mathcal{D}_v$, then $x \mapsto \Phi_v^s(x)$ is a global diffeomorphism of \mathcal{D}_v onto \mathcal{D}_v)

For every fixed $t \in I$, every x in a neighborhood of x^0 , and every $\xi \in T\mathcal{M}_x$ by $(\Phi_v^t)_* \xi$ we denote the image of ξ under the tangent map of the diffeomorphism $y \mapsto \Phi_v^t(y)$ at point x . (Actually, $(\Phi_v^t)_* \xi$ depends on two arguments ξ and x , and we should write $(\Phi_v^t)_*(x, \xi)$, in general, but in our case it will be always clear at which point $x \in \mathcal{M}$ we consider the tangent map, which is why we write $(\Phi_v^t)_* \xi$ without any ambiguity.)

In addition, if V is a vector space, then, for $A \subset V$, and $B \subset V$, we denote by $A + B$ the set $\{x + y \mid x \in A, y \in B\}$ (in our situation V will be $T\mathcal{M}_x$ for some smooth manifold \mathcal{M} and some $x \in \mathcal{M}$).

If $\Delta(\cdot)$ is a smooth integrable distribution on \mathcal{M} , (which means that the dimension $\dim \Delta(x)$ equals $k \leq n$ for some fixed $k = 1, \dots, n$, and for all $x \in \mathcal{M}$, and $\Delta(\cdot)$ is involutive at each point $x \in \mathcal{M}$) then, for each $x^0 \in \mathcal{M}$, we can consider its orbit, or the maximal integral manifold $\mathcal{M}_\Delta(x^0)$ defined as the set of all points $y \in \mathcal{M}$ given by

$$y = (\Phi_{v_1}^{t_1} \circ \Phi_{v_2}^{t_2} \circ \dots \circ \Phi_{v_N}^{t_N})(x^0) \quad (1)$$

with arbitrary $N \geq 1$, arbitrary $t_i \in \mathbf{R}$, $i = 1, \dots, N$, and arbitrary smooth vector fields $v_i(\cdot)$ such that, for every $i = 1, \dots, N$, and every $x \in \mathcal{D}_{v_i}$ we have $v_i(x) \in \Delta(x)$. Also we will use a more brief form of (1):

$$y = \Phi_v^T(x^0) \quad \text{with } T = (t_1, \dots, t_N), \quad v = (v_1, \dots, v_N). \quad (2)$$

By $\Phi_v^{-T}(\cdot)$ we denote the inverse diffeomorphism, i.e. $(\Phi_{v_N}^{-t_N} \circ \Phi_{v_{N-1}}^{-t_{N-1}} \circ \dots \circ \Phi_{v_1}^{-t_1})(\cdot)$

We write by definition $v(\cdot) \in \Delta(\cdot)$ iff $v(x) \in \Delta(x)$ for each $x \in \mathcal{D}_v$. Let us recall that, if $v(\cdot)$, and $w(\cdot)$ are smooth vector fields defined on some open subset $\mathcal{D} \subset \mathcal{M}$, then by $[v, w](\cdot)$ we denote their Lie bracket defined (in any coordinates) as $[v, w](x) = \frac{\partial w}{\partial x} v - \frac{\partial v}{\partial x} w$. Finally, for $A \subset \mathcal{M}$, we denote by \overline{A} the closure of A in \mathcal{M} .

3. Main result.

We consider a control system

$$\dot{x} = a(x) + \beta(x, u)b(x) \quad (3)$$

with states $x \in \mathcal{M}$, and controls $u \in \mathbf{R}^1$, where \mathcal{M} is a simply connected manifold, $a(\cdot)$, $b(\cdot)$, are smooth vector fields (of class C^{n+1} at least) on \mathcal{M} , and $\beta(\cdot, \cdot)$ is a smooth (of class C^{n+1}) scalar function on \mathcal{M} . Next we suppose that $\mathcal{M} = \mathbf{R}^n$ just to make the arguments clearer, however our technique works for arbitrary simply connected manifold \mathcal{M} . We assume that $a(\cdot)$, $b(\cdot)$, and $\beta(\cdot, \cdot)$ satisfy the following conditions

(A) For each $x \in \mathcal{M}$, we have $b(x) \neq 0$, and $\beta(x, \mathbf{R}^1) = \mathbf{R}^1$. In other words, the set $\Delta_0(x) := \{\beta(x, u)b(x) \mid u \in \mathbf{R}^1\}$ is a 1-dimensional subspace of $T\mathcal{M}_x = \mathbf{R}^n$ for each $x \in \mathcal{M}$.

(Of course, the distribution $x \mapsto \Delta_0(x)$ is integrable in the whole \mathcal{M} , and, for each $x^0 \in \mathcal{M}$, the corresponding maximal integral manifold of $\Delta_0(\cdot)$ is the (maximal) trajectory $t \mapsto \Phi_b^t(x^0)$).

(B) Let k be in $\{1, \dots, n-1\}$. Assume that the distribution $\mathcal{M} \ni x \mapsto \Delta_{k-1}(x) \subset T\mathcal{M}_x = \mathbf{R}^n$ is already constructed: for $k=1$, see condition (A), for $k \geq 1$, we define $\Delta_k(\cdot)$ by induction as below. Then:

(B1) We require that $x \mapsto \Delta_{k-1}(x)$ is of rank k for every $x \in \mathcal{M}$, and is involutive at each $x \in \mathcal{M}$.

(B2) Given $x^0 \in \mathcal{M}$, by $\Delta_k(x^0)$ denote the set

$$\Delta_k(x^0) := \Delta_{k-1}(x^0) + \{(\Phi_v^T)_* a(\Phi_v^{-T}(x^0)) - a(x^0) \mid \forall N \geq 1 \quad \forall T = (t_1, \dots, t_N) \\ \forall v(\cdot) = (v_1(\cdot), \dots, v_N(\cdot)) \text{ such that } v_i(x) \in \Delta_{k-1}(x) \text{ for all } x \in \mathcal{D}_{v_i}, i=1, \dots, N\} \quad (4)$$

We require that, for each $x^0 \in \mathcal{M}$, the set $\Delta_k(x^0)$ is a $(k+1)$ - dimensional subspace of $T\mathcal{M}_{x^0} = \mathbf{R}^n$ and that the distribution $x \mapsto \Delta_k(x)$ is involutive (for all $k = 0, \dots, n-1$, as we mentioned before.)

We emphasize that, for each fixed x^0 , we obtain $\Delta_k(x^0)$ in (4) by taking all admissible $v_i(\cdot)$ from Δ_{k-1} (i.e., along the maximal integral manifold $\mathcal{M}_{\Delta_{k-1}}(x^0)$ defined in (1)).

Let us remark that conditions (A), (B1), (B2) are global analog of the conditions from [12], [11].

If a smooth system $\dot{x} = f(x, u)$ is locally feedback equivalent to the triangular form, then (see [16]) $f(\cdot, \cdot)$ have (locally) the form (3): $f(x, u) = a(x) + \beta(x, u)b(x)$ with some smooth vector fields $a(\cdot)$, $b(\cdot)$, $b(x) \neq 0$ and with some smooth scalar function $\beta(\cdot, \cdot)$, regardless of whether this triangular form satisfies the regularity conditions $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, or we deal with the singular case.

Furthermore, any triangular system

$$\begin{cases} \dot{x}_i = f_i(x_1, \dots, x_{i+1}), & i = 1, \dots, n-1; \\ \dot{x}_n = f_n(x_1, \dots, x_n, u); \end{cases} \quad x = (x_1, \dots, x_n)^T \in \mathbf{R}^n, \quad u \in \mathbf{R}^1 \quad (5)$$

such that f_i are smooth, and $f_i(x_1, \dots, x_i, \mathbf{R}^1) = \mathbf{R}^1$, for all $i = 1, \dots, n$ and all $(x_1, \dots, x_i) \in \mathbf{R}^i$ (see [20]) satisfies our conditions (A), (B1), (B2)

Conversely, assume that system (3) satisfies (A), (B1), (B2). Pick any $x^0 \in \mathcal{M}$, and let $\zeta = \varphi(x) : U(x^0) \subset \mathcal{M} \rightarrow V(\varphi(x^0)) \subset \mathbf{R}^n$ ($\zeta_i = \varphi_i(x)$, $i = 1, \dots, n$) be a diffeomorphism of a neighborhood $U(x^0)$ of x^0 onto a neighborhood $V(\varphi(x^0))$ of $\varphi(x^0)$.

Definition 3.1 We say that coordinates ζ_i are **canonical** for system (3), or the map $x \mapsto \zeta = \varphi(x)$ defines canonical coordinates for system (3) (or canonical coordinates for the corresponding sequence of nested regular integrable distributions $\Delta_0(\cdot), \dots, \Delta_{n-1}(\cdot)$) in $U(x^0)$, iff, for each $k = 0, \dots, n-2$, the set $\varphi^{-1}(D_k)$ with

$$D_k := \{(\zeta_1, \dots, \zeta_n) \in V(\varphi(x^0)) \mid \zeta_i = \text{const}, \quad i = 1, \dots, n-k-1\}$$

is an integral manifold of Δ_k in $U(x^0)$.

Equivalently, coordinates $y_i = \varphi_i(x)$ $i = 1, \dots, n$, are canonical for system (3) in $U(x^0)$ with small enough $U(x^0)$, iff

$$\Delta_k(x) = \{\xi \in T\mathcal{M}_x = \mathbf{R}^n \mid \langle \nabla \varphi_i(x), \xi \rangle = 0, \quad i = 1, \dots, n-k-1\}, \quad k = 0, 1, \dots, n-2.$$

Remark 3.1. It is easy to prove that, if coordinates $\zeta_i = \varphi_i(x)$ are canonical for (3) in a neighborhood of some $x^0 \in \mathcal{M}$, then, (**locally!**) in some neighborhood of x^0 , this change of coordinates $\zeta_i = \varphi_i(x)$ brings the dynamics of (3) to the following triangular form

$$\left\{ \begin{pmatrix} \dot{\zeta}_1 \\ \vdots \\ \dot{\zeta}_{n-2} \\ \dot{\zeta}_{n-1} \\ \dot{\zeta}_n \end{pmatrix} = \begin{pmatrix} f_1(\zeta_1, \zeta_2) \\ \vdots \\ f_{n-2}(\zeta_1, \dots, \zeta_{n-1}) \\ f_{n-1}(\zeta_1, \dots, \zeta_n) \\ f_n(\zeta_1, \dots, \zeta_n) \end{pmatrix} + \tilde{\beta}(\zeta, u) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ g_n(\zeta_1, \dots, \zeta_n) \end{pmatrix} \right. \quad (6)$$

where $g_n(\zeta) \neq 0$ in some neighborhood of $\varphi(x^0)$ (but, of course the regularity conditions $|\frac{\partial f_i}{\partial \zeta_{i+1}}| \neq 0$ do not hold, in general). However, this is true only locally, as we can learn from the following example.

Example 3.1 Consider the vector fields $v_1(\cdot)$ and $v_2(\cdot)$ in \mathbf{R}^2 given by $v_1(x) = (\cos x_1, -\sin x_1)^T$, and $v_2(x) = (\sin x_1, \cos x_1)^T$. Put $x_N^0 = (\pi N, 0) \in \mathbf{R}^2$ for all $N \in \mathbf{Z}$. For each $N \in \mathbf{Z}$, the map given by $\mathbf{R}^2 \ni (z_1, z_2) \mapsto (\Phi_{v_2}^{z_2} \circ \Phi_{v_1}^{z_1})(x_N^0)$ is a diffeomorphism of \mathbf{R}^2 onto $]\pi(N-1), \pi(N+1)[\times \mathbf{R}$ (we do not solve the corresponding differential equations explicitly just because the solution can not be written out as a combination of standard functions). Fix any $z_1 > 0$. Denote by $A(z_1) \in \mathbf{R}^2$ the intersection of the trajectory $z_2 \mapsto (\Phi_{v_2}^{z_2} \circ \Phi_{v_1}^{z_1})(x_0^0)$ with $\{\frac{\pi}{2}\} \times \mathbf{R}$; then from the symmetry of the curve $z_2 \mapsto (\Phi_{v_2}^{z_2} \circ \Phi_{v_1}^{z_1})(x_0^0)$ w.r.t. $\{\frac{\pi}{2}\} \times \mathbf{R}$ we obtain that there is a unique solution $(\tilde{z}_2(z_1), \tilde{z}_1(z_1))$ of $(\Phi_{v_2}^{\tilde{z}_2(z_1)} \circ \Phi_{v_1}^{z_1})(x_0^0) = \Phi_{v_1}^{\tilde{z}_1(z_1)}(x_1^0)$ w.r.t. $(\tilde{z}_2(z_1), \tilde{z}_1(z_1))$;

$$\tilde{z}_1(z_1) = z_1; \quad A(z_1) = \left(\Phi_{v_2}^{\frac{\tilde{z}_2(z_1)}{2}} \circ \Phi_{v_1}^{z_1} \right)(x_0^0); \quad (7)$$

and there is a unique solution $(z_2^*(z_1), z_1^*(z_1))$ of the nonlinear equation $(\Phi_{v_2}^{z_2^*(z_1)} \circ \Phi_{v_1}^{z_1^*(z_1)})(x_1^0) = \Phi_{v_1}^{z_1^*(z_1)}(x_0^0)$ w.r.t. $(z_2^*(z_1), z_1^*(z_1))$, and

$$z_1^*(z_1) = z_1; \quad z_2^*(z_1) = -\tilde{z}_1(z_1); \quad A(z_1) = \left(\Phi_{v_2}^{\frac{z_2^*(z_1)}{2}} \circ \Phi_{v_1}^{z_1} \right)(x_1^0); \quad (8)$$

(in addition, the image of the curve $\mathbf{R} \ni z_2 \mapsto (\Phi_{v_2}^{z_2} \circ \Phi_{v_1}^{z_1})(x_1^0)$ coincides with the image of $\mathbf{R} \ni z_2 \mapsto (\Phi_{v_2}^{z_2} \circ \Phi_{v_1}^{z_1})(x_0^0)$).

Similarly, for any fixed $z_1 < 0$, there is a unique solution $(\hat{z}_2(z_1), \hat{z}_1(z_1))$ of the equation $(\Phi_{v_2}^{\hat{z}_2(z_1)} \circ \Phi_{v_1}^{z_1})(x_0^0) = \Phi_{v_1}^{\hat{z}_1(z_1)}(x_{-1}^0)$ w.r.t. (\hat{z}_2, \hat{z}_1) and

$$\hat{z}_1(z_1) = z_1; \quad \hat{z}_2(z_1) = \tilde{z}_2(-z_1) \quad (9)$$

(Of course, for $z_1 < 0$, and for the left half-plane, we could write the equalities which are similar to (8), but we omit that).

By definition, we put

$$\psi(z_1) = \begin{cases} \hat{z}_2(z_1) & \text{if } z_1 < 0 \\ +\infty & \text{if } z_1 = 0 \\ \tilde{z}_2(z_1) & \text{if } z_1 > 0 \end{cases} : \mathbf{R} \rightarrow]0, +\infty[$$

and consider the triangular system

$$\begin{cases} \dot{z}_1 = f_1(z_1, z_2) \\ \dot{z}_2 = u \end{cases} \quad (10)$$

with states $(z_1, z_2) \in \mathbf{R}^2$ and controls $u \in \mathbf{R}^1$, where $f_1(z_1, z_2)$ is given by

$$f_1(z_1, z_2) = \begin{cases} z_2^5 \sin z_2 & \text{if } z_2 \leq 0 \\ 0 & \text{if } 0 < z_2 \leq \psi(z_1) \\ (z_2 - \psi(z_1))^5 \sin(z_2 - \psi(z_1)) & \text{if } z_2 > \psi(z_1) \end{cases} : \mathbf{R} \rightarrow]0, +\infty[$$

(if $z_1=0$, then $\psi(z_1)=+\infty$, and $f_1(0, z_2)=0$ for all $z_2 \geq 0$, by definition.) By definition, put

$$f(z) := \begin{pmatrix} f_1(z_1, z_2) \\ 0 \end{pmatrix}, \quad g(z) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \phi_N(z) = (\Phi_{v_2}^{z_2} \circ \Phi_{v_1}^{z_1})(x_N^0), \quad z = (z_1, z_2) \in \mathbf{R}^2,$$

and define in $]-\pi, \pi[\times \mathbf{R} \subset \mathbf{R}_x^2$ the vector fields $A(\cdot)$, and $B(\cdot)$ by $A = (\phi_0)_* f$, $B = (\phi_0)_* g$. Then the system

$$\dot{x} = A(x) + B(x)u \quad (11)$$

is well-defined in $]-\pi, \pi[\times \mathbf{R}$, and satisfies conditions (A), (B1), (B2). Finally, using the maps $\phi_1(\cdot)$, $\phi_{-1}(\cdot)$, and the above-mentioned symmetry (see (7),(8),(9)), we can easily extend vector fields $A(\cdot)$ and $B(\cdot)$ onto $]-2\pi, +2\pi[\times \mathbf{R}$, then onto $]-3\pi, +3\pi[\times \mathbf{R}$, and eventually onto $\mathbf{R} \times \mathbf{R}$ so that (A), (B1), (B2) hold in \mathbf{R}^2 . Of course, system (11) is not globally feedback equivalent to a triangular system of form (6) in the whole \mathbf{R}^2 (for instance, because it is globally equivalent to (10) in $]-\pi, \pi[\times \mathbf{R}$). On the other hand, system (11) satisfies conditions (A), (B1), (B2) in the whole \mathbf{R}^2 by the construction.

Our main result is as follows.

Theorem 3.1 *Assume that vector fields $a(\cdot)$, $b(\cdot)$ and function $\beta(\cdot, \cdot)$ are of class C^{n+1} , and satisfy conditions (A), (B1), (B2). Then system (3) is globally controllable (in the whole \mathcal{M}) in any time $[t_0, T]$.*

The goal of this paper is to prove theorem 3.1.

4. The reduction of the main result to a "back-stepping" procedure.

As we see from example 3.1, system (3) is not globally feedback equivalent to a system from [20] in general, which is why the technique developed in [19],[20] should be at least revised essentially. However, if we want to follow this pattern, we must first pick a point $(x^*, u^*) \in \mathcal{M} \times \mathbf{R}^1$ around which system (3) is regular, and (locally!) feedback linearizable. Using conditions (A), and (B), we easily get the existence of a point $x^* \in \mathcal{M}$ such that

$$\{b(x), [a, b](x), \dots, (\text{ad}_a^{i-1} b)(x)\} \quad \text{is a basis of } \Delta_{i-1}(x)$$

$$\text{for all } x \in W(x^*), \quad i = 1, \dots, n \quad (12)$$

for some neighborhood $W(x^*)$ of x^* in \mathcal{M} .

Pick any $t_1 \in]t_0, T[$. In order to prove theorem 3.1, it suffices to show that we can steer any initial state x^0 into x^* in time $J := [t_0, t_1]$, and that we can steer x^* into any terminal state x^T in time $I := [t_1, T]$ w.r.t.(3). Next, we prove the second statement only, the proof of the first one being similar. This statement, in turn, follows from the following theorem 4.1.

Theorem 4.1. *Let p be in $\{1, \dots, n-1\}$. Assume that, for every $x^T \in \mathcal{M}$, there exist a curve $t \mapsto z(t)$ of class $C^1(I; \mathcal{M})$, and a map $(t, x) \mapsto \varphi(t, x) = (\varphi_1(t, x), \dots, \varphi_n(t, x)) \in \mathbf{R}^n$ of class C^1 defined in some neighborhood $E \subset I \times \mathcal{M}$ of the curve $\{(t, z(t)) \in I \times \mathcal{M} \mid t \in I\}$ such that:*

1) *for each fixed $t \in I$, the map $x \mapsto (\varphi_1(t, x), \dots, \varphi_n(t, x))$ defines canonical coordinates for system (3) in the corresponding neighborhood $E_t := \{x \in \mathcal{M} \mid (t, x) \in E\}$ of $z(t) \in \mathcal{M}$.*

2) *$z(\cdot)$, and $\varphi(\cdot, \cdot)$ satisfy the equalities*

$$\begin{aligned} \frac{\partial \varphi_i(t, z(t))}{\partial x} \dot{z}(t) &= \frac{\partial \varphi_i(t, z(t))}{\partial x} a(z(t)), & i = 1, \dots, p-1, \quad t \in I; \\ \frac{d\varphi_i(t, z(t))}{dt} &= 0, & i = 1, \dots, n, \quad t \in I; \end{aligned} \quad (13)$$

(For $p=1$, (13) has the form $\frac{d\varphi_i(t, z(t))}{dt} = 0$, $i = 1, \dots, n$, $t \in I$, by definition)

3) *$z(t_1) = x^*$, $z(T) = x^T$, and $\frac{\partial \varphi_p}{\partial x}(t_1, x^*) \dot{z}(t_1) = \frac{\partial \varphi_p}{\partial x}(t_1, x^*) a(x^*)$.*

Then, for every $x^T \in \mathcal{M}$, there exist a curve $t \mapsto y(t)$ of class $C^1(I, \mathcal{M})$, and a map $(t, x) \mapsto \psi(t, x) = (\psi_1(t, x), \dots, \psi_n(t, x)) \in \mathbf{R}^n$ of class C^1 defined in some neighborhood $G \subset I \times \mathcal{M}$ of the curve $\{(t, y(t)) \in I \times \mathcal{M} \mid t \in I\}$ such that

4) *For each fixed $t \in I$, the map $x \mapsto (\psi_1(t, x), \dots, \psi_n(t, x))$ defines canonical coordinates for system (3) in the neighborhood $G_t := \{x \in \mathcal{M} \mid (t, x) \in G\}$ of $y(t) \in \mathcal{M}$.*

5) *$y(\cdot)$ and $\psi(\cdot, \cdot)$ satisfy the equalities*

$$\begin{aligned} \frac{\partial \psi_i(t, y(t))}{\partial x} \dot{y}(t) &= \frac{\partial \psi_i(t, y(t))}{\partial x} a(y(t)), & i = 1, \dots, p, \quad t \in I; \\ \frac{d\psi_i(t, y(t))}{dt} &= 0, & i = 1, \dots, n, \quad t \in I; \end{aligned} \quad (14)$$

6) *$y(t_1) = x^*$, $y(T) = x^T$, and $\frac{\partial \psi_{p+1}}{\partial x}(t_1, x^*) \dot{y}(t_1) = \frac{\partial \psi_{p+1}}{\partial x}(t_1, x^*) a(x^*)$*

Let us show that theorem 4.1 implies theorem 3.1. Indeed, for $p = 1$, the construction of $z(\cdot)$, and $\varphi(\cdot, \cdot)$ such that conditions 1), 2), 3) of theorem 4.1 hold is straightforward. Let $\zeta = \tilde{\varphi}(x)$ ($\zeta_i = \tilde{\varphi}_i(x)$, $i = 1, \dots, n$) be any canonical coordinates for system (3) in a neighborhood of x^* , and let (6) be the dynamics of (3) in the coordinates $(\zeta_1, \dots, \zeta_n)$. Given an arbitrary $x^T \in \mathcal{M}$, pick any $z(\cdot) \in C^1(I; \mathcal{M})$ such that $z(t_1) = x^*$, $z(T) = x^T$. Since the definition of the Lie derivative does not depend on local coordinates, the condition $\frac{\partial \tilde{\varphi}_1}{\partial x}(t_1, x^*) \dot{z}(t_1) = \frac{\partial \tilde{\varphi}_1}{\partial x}(t_1, x^*) a(x^*)$ is equivalent to the equality $\dot{\zeta}_1(t_1) = f_1(\tilde{\zeta}_1^*, \tilde{\zeta}_2^*)$ (where $\zeta(t) = \tilde{\varphi}(z(t))$, $\zeta_i(t) := \tilde{\varphi}_i(z(t))$, $\tilde{\zeta}_i^* = \tilde{\varphi}_i(x^*)$, $i = 1, \dots, n$), and does not depend on the choice of canonical coordinates around x^* . Therefore, we can easily choose $z(\cdot) \in C^1(I; \mathcal{M})$

such that the condition $\frac{\partial \tilde{\varphi}_1}{\partial x}(t_1, x^*) \dot{z}(t_1) = \frac{\partial \tilde{\varphi}_1}{\partial x}(t_1, x^*) a(x^*)$ holds along with the equalities $z(t_1)=x^*$, $z(T)=x^T$ from the very beginning. Then $z(\cdot)$ satisfies condition 3) with $p=1$ for every function $(t, x) \mapsto \varphi(t, x) \in \mathbf{R}^n$ defined in a neighborhood of $\{(t, z(t)) \mid t \in I\}$, and satisfying conditions 1), 2) of theorem 4.1. In order to construct $\varphi(\cdot, \cdot)$, consider vector fields $v_1(\cdot), \dots, v_n(\cdot)$ on \mathcal{M} such that $\Delta_i(x) = \text{span}\{v_{n-i}(x), v_{n-i+1}(x), \dots, v_n(x)\}$ for all $x \in \mathcal{M}$, $i=0, \dots, n-1$. (Since \mathcal{M} is simply connected, it is orientable as well as $\Delta_i(\cdot)$, $i = 0, \dots, n-1$, and such vector fields do exist). Then, for each $\xi \in \mathcal{M}$, the map $(t_1, \dots, t_n) \mapsto (\Phi_{v_n}^{t_n} \circ \Phi_{v_{n-1}}^{t_{n-1}} \circ \dots \circ \Phi_{v_1}^{t_1})(\xi)$ is a diffeomorphism of some (small) neighborhoods $\mathcal{B}_\xi(0)$ and $U(\xi)$ of $0 \in \mathbf{R}^n$ and $\xi \in \mathcal{M}$ respectively. Let $x \mapsto \phi(\xi, x)$ ($t_i = \phi_i(\xi, x)$, $i=1, \dots, n$) be the inverse diffeomorphism of $U(\xi)$ onto $\mathcal{B}_\xi(0)$. For any fixed (t_1^0, \dots, t_n^0) , the map $(t_{i+1}, \dots, t_n) \mapsto (\Phi_{v_n}^{t_n} \circ \Phi_{v_{n-1}}^{t_{n-1}} \circ \dots \circ \Phi_{v_{i+1}}^{t_{i+1}} \circ \Phi_{v_i}^{t_i^0} \circ \dots \circ \Phi_{v_1}^{t_1^0})(\xi)$ defines the integral manifold of the distribution $\Delta_{n-i-1}(\cdot)$ in $U(\xi)$, $i=1, \dots, n-1$. Therefore, for every fixed $\xi \in \mathcal{M}$, the map $x \mapsto \phi(\xi, x)$ ($t_i = \phi_i(\xi, x)$, $i=1, \dots, n$) defines canonical coordinates (t_1, \dots, t_n) for system (3) in some neighborhood $U(\xi)$ of ξ . Taking into account that $\phi(\xi, \xi) = 0$ for all $x \in \mathcal{M}$, we obtain that the map $x \mapsto \varphi(t, x)$ defined by $\varphi(t, x) = \phi(z(t), x)$ in some neighborhood of $\{(t, z(t)) \mid t \in I\}$ and the curve $z(\cdot) \in C^1(I; \mathcal{M})$ satisfy conditions 1), 2), 3) of theorem 4.1 with $p=1$.

Then, using theorem 4.1, and induction over $p = 1, 2, \dots, n-1$, we get the existence of a curve $y(\cdot) \in C^1(I; \mathcal{M})$ and a map $x \mapsto \psi(t, x) = (\psi_1(t, x), \dots, \psi_n(t, x)) \in \mathbf{R}^n$ of class C^1 in a neighborhood $G \subset I \times \mathcal{M}$ of $\{(t, y(t)) \mid t \in I\}$ such that conditions 4), 5), 6) of theorem 4.1 hold with $p=n-1$, which implies, in particular that $y(t_1)=x^*$, $y(T)=x^T$, and

$$\frac{\partial \psi_i(t, y(t))}{\partial x} \dot{y}(t) = \frac{\partial \psi_i(t, y(t))}{\partial x} a(y(t)), \quad i = 1, \dots, n-1, \quad t \in I; \quad (15)$$

Since $(\psi_1(t, \cdot), \dots, \psi_n(t, \cdot))$ are canonical coordinates in some neighborhood of $y(t)$ for each $t \in I$, we obtain from condition (A):

$$\begin{aligned} \frac{\partial \psi_i(t, y(t))}{\partial x} b(y(t)) &= 0, \quad i = 1, \dots, n-1, \\ \frac{\partial \psi_n(t, y(t))}{\partial x} b(y(t)) &\neq 0, \quad \text{for all } t \in I \end{aligned} \quad (16)$$

Therefore, (15) implies that

$$\begin{aligned} \frac{\partial \psi_i(t, y(t))}{\partial x} \dot{y}(t) &= \frac{\partial \psi_i(t, y(t))}{\partial x} (a(y(t)) + \beta(y(t), u(t)) b(y(t))), \quad t \in I \\ i &= 1, \dots, n-1, \quad \text{for each } u(\cdot) \in L_\infty(I; \mathbf{R}^1) \end{aligned}$$

Then using a modification of the well-known Filippov lemma, condition (A), and (16) (a similar argument can be found in [17]), we get the existence of $u(\cdot) \in L_\infty(I; \mathbf{R}^1)$ such that

$$\frac{\partial \psi_n(t, y(t))}{\partial x} \dot{y}(t) = \frac{\partial \psi_n(t, y(t))}{\partial x} (a(y(t)) + \beta(y(t), u(t)) b(y(t))), \quad \text{a. e. on } I,$$

which yields

$$\frac{\partial \psi(t, y(t))}{\partial x} \dot{y}(t) = \frac{\partial \psi(t, y(t))}{\partial x} (a(y(t)) + \beta(y(t), u(t)) b(y(t))), \quad \text{a. e. on } I$$

Since the Jakoby matrix $\frac{\partial \psi(t, y(t))}{\partial x}$ is invertible for every $t \in I$, the last equality is equivalent to

$$\dot{y}(t) = a(y(t)) + \beta(y(t), u(t))b(y(t)), \quad \text{a. e. on } I.$$

Therefore, $u(\cdot)$ steers x^* into x^T in time $I = [t_1, T]$ w.r.t (3). The proof of the fact that x^0 can be steered into x^* in time $J = [t_0, t_1]$ is similar. Thus our main goal is to prove theorem 4.1.

5. Proof of theorem 4.1.

Next we always treat \mathcal{M} as \mathbf{R}^n , and always write \mathbf{R}^n instead of \mathcal{M} just to simplify the notation, and to make the arguments clearer. For every $\varepsilon > 0$, every $\zeta \in \mathbf{R}^p$, and every $x \in \mathbf{R}^n$, we denote by $B_\varepsilon(\zeta)$, and $\Omega_\varepsilon(x)$ the open balls in \mathbf{R}^p and in \mathbf{R}^n respectively given by $B_\varepsilon(\zeta) := \{\bar{\zeta} \in \mathbf{R}^p \mid |\bar{\zeta} - \zeta| < \varepsilon\}$; $\Omega_\varepsilon(x) := \{\bar{x} \in \mathbf{R}^n \mid |\bar{x} - x| < \varepsilon\}$, where $|\cdot|$ is the standard norm generated by the standard scalar products of \mathbf{R}^p and \mathbf{R}^n respectively.

Take an arbitrary $p \in \{1, \dots, n-1\}$, and an arbitrary $x^T \in \mathbf{R}^n$. Assume that a curve $z(\cdot) \in C^1(I; \mathbf{R}^n)$, and a map $(t, x) \mapsto \varphi(t, x) = (\varphi_1(t, x), \dots, \varphi_n(t, x)) \in \mathbf{R}^n$, which is of class C^1 in some neighborhood $E \subset I \times \mathbf{R}^n$ of the set $\{(t, z(t)) \in I \times \mathbf{R}^n \mid t \in I\}$, satisfy conditions 1), 2), 3) of theorem 4.1. Let $\zeta = \phi(x)$ ($\zeta_i = \phi_i(x)$, $i=1, \dots, n$) be some fixed canonical coordinates for system (3) in some small neighborhood $U(x^*) \subset W(x^*)$ of x^* , where $W(x^*)$ is defined in (12) (for instance, we may put $\phi(x) := \varphi(t_1, x)$ - see conditions 1), 3) of theorem 4.1), and let (6) be the dynamics of (3) in the local coordinates $(\zeta_1, \dots, \zeta_n)$ (in the neighborhood $U(x^*)$). Choose $\bar{\sigma} > 0$ ($\bar{\sigma} < T - t_1$) such that $z(t) \in U(x^*)$ for all $t \in [t_1, t_1 + \bar{\sigma}]$, and put by definition $D := \phi(U(x^*))$; $\zeta^* := \phi(x^*)$; $\zeta_i^* := \phi_i(x^*)$, $i=1, \dots, n$; $\zeta^*(t) := \phi(z(t))$, $\zeta_i^*(t) := \phi_i(z(t))$, $i=1, \dots, n$, for all $t \in [t_1, t_1 + \bar{\sigma}]$.

Without loss of generality, we assume that

$$D = \{(\zeta_1, \dots, \zeta_n) \in \mathbf{R}^n \mid |\zeta_k| < \sigma_k, \quad k = 1, \dots, n\}$$

with some $\sigma_k > 0$, $k=1, \dots, n$, and that every integral manifold of each $\Delta_i(\cdot)$ ($i=0, \dots, n-2$) in $D = \phi(U(x^*))$ is equal to

$$\{(\zeta_1, \dots, \zeta_n) \in \mathbf{R}^n \mid \zeta_k = \zeta_k^0, \quad k = 1, \dots, n-i-1; \quad |\zeta_k| < \sigma_k, \quad k=n-i, \dots, n\}$$

with some ζ_k^0 , $k=1, \dots, n-i-1$ such that $|\zeta_k^0| < \sigma_k$.

By the construction (see conditions 2), and 3) of theorem 4.1), we have

$$\begin{aligned} \dot{\zeta}_i^*(t) &= f_i(\zeta_1^*(t), \dots, \zeta_{i+1}^*(t)), \quad i = 1, \dots, p-1; \quad t \in [t_1, t_1 + \bar{\sigma}] \\ \zeta_i^*(t_1) &= \zeta_i^*, \quad i = 1, \dots, n; \quad \dot{\zeta}_p^*(t_1) = f_p(\zeta_1^*, \dots, \zeta_p^*, \zeta_{p+1}^*). \end{aligned}$$

In addition,

$$\frac{\partial f_p}{\partial \zeta_{p+1}}(\zeta_1^*, \dots, \zeta_p^*, \zeta_{p+1}^*) \neq 0$$

- see (12). Therefore, there exist $\sigma \in]0, \bar{\sigma}[$ and $w(\cdot) \in C([t_1, t_1 + \sigma]; \mathbf{R}^1)$ such that $w(t_1) = \zeta_{p+1}^*$, and

$$|w(t)| < \sigma_{p+1}, \quad \text{i. e.,} \quad (\zeta_1^*(t), \dots, \zeta_p^*(t), w(t), \zeta_{p+2}^*(t), \dots, \zeta_n^*(t)) \in D$$

$$\text{for all } t \in [t_1, t_1 + \sigma] \quad (17)$$

and

$$\dot{\zeta}_p^*(t) = f_p(\zeta_1^*(t), \dots, \zeta_p^*(t), w(t)), \quad t \in [t_1, t_1 + \sigma] \quad (18)$$

For any $\tilde{\zeta}_{p+1}(\cdot) \in C([t_1, t_1 + \sigma]; \mathbf{R}^1)$ such that

$$|\tilde{\zeta}_{p+1}(t)| < \sigma_{p+1}, \quad \text{i. e., } \{(\zeta_1, \dots, \zeta_p, \tilde{\zeta}_{p+1}(t), \zeta_{p+2}^*(t), \dots, \zeta_n^*(t)) \mid \zeta_i \in \mathbf{R}^1, \quad i=1, \dots, p\} \cap D \neq \emptyset$$

for all $t \in [t_1, t_1 + \sigma]$, we denote by $t \mapsto \eta(t, \tilde{\zeta}_{p+1}(\cdot))$ the (maximal) trajectory of the p -dimensional control system

$$\dot{\zeta}_i(t) = f_i(\zeta_1(t), \dots, \zeta_{i+1}(t)), \quad i = 1, \dots, p; \quad t \in [t_1, t_1 + \sigma] \quad (19)$$

(with states $(\zeta_1, \dots, \zeta_p)$) with the control $\zeta_{p+1}(\cdot) = \tilde{\zeta}_{p+1}(\cdot)$, and with the initial condition $\zeta_i(t_1) = \zeta_i^*$, $i=1, \dots, p$.

Since $|\frac{\partial f_i}{\partial \zeta_{i+1}}| \neq 0$, $i=1, \dots, n$ in D (see (12)), the linearization of (19) around $(\zeta_1^*(\cdot), \dots, \zeta_p^*(\cdot), w(\cdot))$ given by

$$\dot{\chi}_i(t) = \sum_{j=1}^{i+1} \frac{\partial f_i}{\partial \zeta_j}(\zeta_1^*(t), \dots, \zeta_{i+1}^*(t)) \chi_j(t), \quad i = 1, \dots, p; \quad t \in [t_1, t_1 + \sigma] \quad (20)$$

with states $(\chi_1, \dots, \chi_p) \in \mathbf{R}^p$ and controls $\chi_{p+1} \in \mathbf{R}^1$ is completely controllable; therefore there exist p controls $w_i(\cdot) \in C^1([t_1, t_1 + \sigma]; \mathbf{R}^1)$, $i = 1, \dots, p$ such that

$$(C_1) \quad \dot{w}_i(t_1) = w_i(t_1) = w_i(t_1 + \sigma) = \dot{w}_i(t_1 + \sigma) = 0, \quad i = 1, \dots, p;$$

(C₂) Each control $w_i(\cdot)$ steers $0 \in \mathbf{R}^p$ into $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbf{R}^p$ (the i -th unit vector of the standard basis in \mathbf{R}^p) in time I w.r.t. (20).

For every $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbf{R}^p$, we define $w_\lambda(\cdot)$ as follows $w_\lambda(t) := w(t) + \sum_{j=1}^p \lambda_j w_j(t)$, $t \in [t_1, t_1 + \sigma]$. Then, $w_\lambda(\cdot)|_{\lambda=0} = w(\cdot)$; $\eta(t, w_\lambda(\cdot)) = (\zeta_1^*(t), \dots, \zeta_p^*(t))$, for all $t \in [t_1, t_1 + \sigma]$, and, therefore, the trajectory $t \mapsto \eta(t, w_\lambda(\cdot))$ is well-defined on $[t_1, t_1 + \sigma]$ for all λ from some small neighborhood of $0 \in \mathbf{R}^p$; $(\eta(t, w_\lambda(\cdot)), w_\lambda(t), \zeta_{p+2}^*(t), \dots, \zeta_n^*(t)) \in D$ for all $t \in [t_1, t_1 + \sigma]$, and all λ in this neighborhood of $0 \in \mathbf{R}^p$; and the map $\lambda \mapsto F(\lambda)$ given by $F(\lambda) := \eta(t_1 + \sigma, w_\lambda(\cdot))$ is well-defined in this neighborhood of $\lambda=0 \in \mathbf{R}^p$.

Furthermore, from (C₂), we get $\frac{\partial F}{\partial \lambda}(0) = (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbf{R}^p$ (the i -th unit vector of the standard canonical basis in \mathbf{R}^p); therefore, there is $\varepsilon > 0$ such that the map $\lambda \mapsto F(\lambda)$ is a well-defined diffeomorphism of $B_\varepsilon(0)$ onto some open neighborhood of $\eta(t_1 + \sigma, w(\cdot)) = (\zeta_1^*(t_1 + \sigma), \dots, \zeta_p^*(t_1 + \sigma))$. Then there exists $\varepsilon_1 > 0$ such that

$$\overline{B_{\varepsilon_1}(\eta(t_1 + \sigma, w(\cdot)))} \subset F(B_\varepsilon(0)) \quad \text{and} \quad \overline{\Omega_{\varepsilon_1}(\zeta^*(t_1 + \sigma))} \subset D.$$

Without loss of generality, we may assume that $\varepsilon > 0$, and $\varepsilon_1 > 0$ are small enough and satisfy the condition:

$$|\alpha \zeta_{p+1}^*(t) + (1 - \alpha) w_\lambda(t) + \xi_{p+1}| < \sigma_{p+1}, \quad \text{whenever } |\lambda| < \varepsilon, \quad 0 \leq \alpha \leq 1,$$

$$|\xi_{p+1}| < \varepsilon_1, \quad t \in [t_1, t_1 + \sigma] \quad (21)$$

Fix any $\varepsilon_2 > 0$ such that

$$\overline{\Omega_{\varepsilon_2}(z(t))} \subset E_t \quad \text{for all } t \in I, \quad (22)$$

where E_t was defined in condition 1) of theorem 4.1, and

$$\phi(\overline{\Omega_{\varepsilon_2}(z(t_1 + \sigma))}) \subset \Omega_{\frac{\varepsilon_1}{2}}(\zeta^*(t_1 + \sigma)) \quad (23)$$

Lemma 5.1. *There are a curve $y(\cdot) \in C^1([t_1 + \sigma, T], \mathbf{R}^n)$, and a map $(t, x) \mapsto \psi(t, x) = (\psi_1(t, x), \dots, \psi_n(t, x)) \in \mathbf{R}^n$ of class C^1 defined in some neighborhood $\tilde{G} \subset [t_1 + \sigma, T] \times \mathbf{R}^n$ of the curve $\{(t, x) \in [t_1 + \sigma, T] \times \mathbf{R}^n \mid x = y(t), t \in [t_1 + \sigma, T]\}$ such that*

1) *For each fixed $t \in [t_1 + \sigma, T]$, the map $x \mapsto (\psi_1(t, x), \dots, \psi_n(t, x))$ defines canonical coordinates for system (3) in the neighborhood $\tilde{G}_t := \{x \in \mathbf{R}^n \mid (t, x) \in \tilde{G}\}$ of $y(t)$.*

2) *$y(\cdot)$ and $\psi(\cdot, \cdot)$ satisfy the equalities*

$$\begin{aligned} \frac{\partial \psi_i(t, y(t))}{\partial x} \dot{y}(t) &= \frac{\partial \psi_i(t, y(t))}{\partial x} a(y(t)), \quad i = 1, \dots, p, \quad t \in [t_1 + \sigma, T]; \\ \frac{d\psi_i(t, y(t))}{dt} &= 0, \quad i = 1, \dots, n, \quad t \in [t_1 + \sigma, T]; \end{aligned}$$

3) *$y(T) = x^T$, and $y(t_1 + \sigma) \in \overline{\Omega_{\varepsilon_2}(z(t_1 + \sigma))}$.*

The proof of lemma 5.1 is given below in section 6. (Of course, it is based on condition (B), and on definition 3.1).

Let us assume that lemma 5.1 is already proved. This allows us to complete the proof of theorem 4.1 as follows. Put

$$\hat{\zeta}(t) := \phi(y(t)), \quad \hat{\zeta}_i(t) := \phi_i(y(t)), \quad i = 1, \dots, n, \quad t \in [t_1 + \sigma, t_1 + \hat{\sigma}], \quad (24)$$

where $\hat{\sigma} \in]\sigma, \bar{\sigma}]$ is such that $y(t) \in U(x^*)$, whenever $t \in [t_1 + \sigma, t_1 + \hat{\sigma}]$. Then, we obtain from (23) and from condition 3) of lemma 5.1

$$\hat{\zeta}(t_1 + \sigma) \in \Omega_{\frac{\varepsilon_1}{2}}(\zeta^*(t_1 + \sigma)). \quad (25)$$

Using (21), and the standard argument based on the Gronwall-Bellmann lemma, and on the Brouwer fixed point theorem - see [24] (and [19], [20] for our case) we get the existence of a control $\hat{w}(\cdot)$ of class $C^1([t_1, t_1 + \sigma]; \mathbf{R}^1)$, ($|\hat{w}(t)| < \sigma_{p+1}$, $t \in [t_1, t_1 + \sigma]$), whereas the norm $\|\hat{w}(\cdot) - w(\cdot)\|_{L_1([t_1, t_1 + \sigma]; \mathbf{R}^1)}$ should be small enough) such that

$$(C_3) \quad \hat{w}(t_1) = \zeta_{p+1}^*, \quad \frac{d\hat{w}}{dt}(t_1) = f_{p+1}(\zeta_1^*, \dots, \zeta_p^*, \zeta_{p+1}^*, \zeta_{p+2}^*) \quad \hat{w}(t_1 + \sigma) = \hat{\zeta}_{p+1}(t_1 + \sigma), \quad \frac{d\hat{w}}{dt}(t_1 + \sigma) = \frac{d\hat{\zeta}_{p+1}}{dt}(t_1 + \sigma).$$

$$(C_4) \quad \text{The map } \lambda \mapsto \hat{F}(\lambda) := \eta(t_1 + \sigma, \hat{w}(\cdot) + \sum_{j=1}^p \lambda_j w_j(\cdot)) \text{ is well-defined for all } \lambda \in B_\varepsilon(0), \text{ and } \overline{B_{\frac{\varepsilon_1}{2}}(\eta(t_1 + \sigma, w(\cdot)))} \subset \hat{F}(B_\varepsilon(0)).$$

Then we obtain from condition (C₄) and from (25) (see [19], [20]) that there exists $\lambda^* = (\lambda_1^*, \dots, \lambda_p^*)$ in $B_\varepsilon(0)$ such that $(\hat{\zeta}_1(t_1 + \sigma), \dots, \hat{\zeta}_p(t_1 + \sigma)) = \eta(t_1 + \sigma, \hat{w}_{\lambda^*}(\cdot))$, where $\hat{w}_{\lambda^*}(\cdot)$

is given by $\hat{w}_{\lambda^*}(t) := \hat{w}(t) + \sum_{j=1}^p \lambda_j^* w_j(t)$, $t \in [t_1, t_1 + \sigma]$. Let us define $y(t)$, and $\bar{\zeta}(t) := \phi(y(t))$ on $[t_1, t_1 + \sigma]$ as follows: by definition, put

$$(\bar{\zeta}_1(t), \dots, \bar{\zeta}_p(t)) := \eta(t, \hat{w}_{\lambda^*}(\cdot)), \quad \bar{\zeta}_{p+1}(t) := \hat{w}_{\lambda^*}(t), \quad t \in [t_1, t_1 + \sigma]; \quad (26)$$

in addition, let $\bar{\zeta}_{p+2}(\cdot), \dots, \bar{\zeta}_n(\cdot)$ be any functions of class $C^1([t_1, t_1 + \sigma]; \mathbf{R}^1)$ such that

$$\bar{\zeta}_i(t_1 + \sigma) = \hat{\zeta}_i(t_1 + \sigma), \quad \frac{d\bar{\zeta}_i}{dt}(t_1 + \sigma) = \frac{d\hat{\zeta}_i}{dt}(t_1 + \sigma), \quad i = p+2, \dots, n, \quad (27)$$

$$\bar{\zeta}_i(t_1) = \zeta_i^*, \quad i = p+2, \dots, n, \quad (28)$$

and such that

$$|\bar{\zeta}_i(t)| < \sigma_i, \quad i = p+2, \dots, n, \text{ i. e., } (\bar{\zeta}_1(t), \dots, \bar{\zeta}_n(t)) \in D \quad \text{for all } t \in [t_1, t_1 + \sigma], \quad (29)$$

and then put:

$$\bar{\zeta}(t) := (\bar{\zeta}_1(t), \dots, \bar{\zeta}_n(t)), \quad y(t) := \phi^{-1}(\bar{\zeta}(t)), \quad \text{for all } t \in [t_1, t_1 + \sigma] \quad (30)$$

Thus, we have constructed $y(\cdot)$ of class $C^1(I; \mathbf{R}^1)$ such that conditions 4), 5), 6) of theorem 4.1 hold with every canonical coordinate functions $\psi_j(t, y)$, $j = 1, \dots, n$. Indeed, the inclusion $y(\cdot) \in C^1(I; \mathbf{R}^n)$ follows from (24), from (30), and from (27),(26) (in addition, we take into account that

$$\bar{\zeta}_{p+1}(t_1 + \sigma) = \hat{\zeta}_{p+1}(t_1 + \sigma), \quad \frac{d\bar{\zeta}_{p+1}}{dt}(t_1 + \sigma) = \frac{d\hat{\zeta}_{p+1}}{dt}(t_1 + \sigma)$$

by (C₁), (C₃), and by (26), and that

$$\begin{aligned} \frac{d\hat{\zeta}_i}{dt}(t_1 + \sigma) &= f_i(\hat{\zeta}_1(t_1 + \sigma), \dots, \hat{\zeta}_{i+1}(t_1 + \sigma)) = f_i(\bar{\zeta}_1(t_1 + \sigma), \dots, \bar{\zeta}_{i+1}(t_1 + \sigma)) = \\ &= \frac{d\bar{\zeta}_i}{dt}(t_1 + \sigma), \quad i = 1, \dots, p \end{aligned}$$

by the construction). Conditions 4), and 5) of theorem 4.1 follow from conditions 1), and 2) of lemma 5.1 respectively (by the construction, (14) is true for all $t \in [t_1, t_1 + \sigma]$, and we can easily construct the appropriate $\psi(t, x)$ for all $t \in I$ following the argument from section 4). The equality $y(T) = x^T$ follows from condition 3) of lemma 5.1; the equality $y(t_1) = x^*$ follows from the definition of $\bar{\zeta}(\cdot)$: indeed, by the definition of $\eta(t, v(\cdot))$, we have

$$(\bar{\zeta}_1(t_1), \dots, \bar{\zeta}_p(t_1)) = \eta(t_1, \hat{w}_{\lambda^*}(\cdot)) = (\zeta_1^*, \dots, \zeta_p^*);$$

conditions (C₃), and (C₁) yield $\bar{\zeta}_{p+1}(t_1) = \hat{w}_{\lambda^*}(t_1) = \zeta_{p+1}^*$; taking into account (28), we obtain $\bar{\zeta}(t_1) = (\zeta_1^*, \dots, \zeta_n^*) = \zeta^*$, which implies that $y(t_1) = \phi^{-1}(\bar{\zeta}(t_1)) = \phi^{-1}(\zeta^*) = x^*$.

Finally, we obtain from (C₃), and from (C₁)

$$\frac{d\hat{w}_{\lambda^*}}{dt}(t_1) = f_{p+1}(\zeta_1^*, \dots, \zeta_p^*, \zeta_{p+1}^*, \zeta_{p+2}^*),$$

which yields: $\frac{\partial \psi_{p+1}}{\partial x}(t_1, x^*) \dot{y}(t_1) = \frac{\partial \psi_{p+1}}{\partial x}(t_1, x^*) a(x^*)$ (because the definition of the Lie derivative does not depend on coordinates). Therefore, $y(\cdot)$, and $\psi(\cdot, \cdot)$ satisfy condition 6) of theorem 4.1 as well. The proof of theorem 4.1 is complete.

6. Proof of lemma 5.1.

Consider the following control system of ordinary differential equations

$$\begin{cases} \frac{\partial \varphi_j}{\partial x}(t, x(t)) \dot{x}(t) = \frac{\partial \varphi_j}{\partial x}(t, x(t)) a(x(t)), & j=1, \dots, p-1, \quad t \in I, \quad (t, x) \in E; \\ \frac{\partial \varphi_p}{\partial t}(t, x(t)) + \frac{\partial \varphi_p}{\partial x}(t, x(t)) \dot{x}(t) = v(t); & t \in I, \quad (t, x) \in E \\ \frac{\partial \varphi_j}{\partial t}(t, x(t)) + \frac{\partial \varphi_j}{\partial x}(t, x(t)) \dot{x}(t) = 0, & j=p+1, \dots, n, \quad t \in I, \quad (t, x) \in E; \end{cases} \quad (31)$$

(If $p = 1$, then the first row is empty by definition) with states $x \in \mathbf{R}^n$, $(t, x) \in E$, and controls $v \in \mathbf{R}^1$. Since the Jakoby matrix $\frac{\partial \varphi}{\partial x}(t, x)$ is invertible for all $(t, x) \in E$ (see condition 1) of theorem 4.1), we can rewrite (31) in its standard form

$$\dot{x}(t) = F(t, x(t), v(t)), \quad t \in I, \quad (t, x) \in E, \quad (32)$$

where $F(\cdot, \cdot, \cdot)$ is given by

$$F(t, x, v) = \left[\frac{\partial \varphi(t, x)}{\partial x} \right]^{-1} \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(t, x) a(x) \\ \vdots \\ \frac{\partial \varphi_{p-1}}{\partial x}(t, x) a(x) \\ v - \frac{\partial \varphi_p}{\partial t}(t, x) \\ -\frac{\partial \varphi_{p+1}}{\partial t}(t, x) \\ \vdots \\ -\frac{\partial \varphi_n}{\partial t}(t, x) \end{pmatrix} \quad (33)$$

Given $(\tau, \tilde{x}) \in E$ and $v(\cdot) \in L_\infty(I; \mathbf{R}^1)$, denote by $t \mapsto x(t, \tau, \tilde{x}, v(\cdot))$ the maximal trajectory of system (31) with the control $v(\cdot)$ and with the initial condition $x(\tau, \tau, \tilde{x}, v(\cdot)) = \tilde{x}$ (of course, $(t, x(t, \tau, \tilde{x}, v(\cdot))) \in E$ for all admissible t). In addition, if $v = v(t, x)$ is a feedback control, which can be time-varying, and even discontinuous, defined in some open subset $\tilde{E} \subset E$, in general, and if $(\tau, \tilde{x}) \in \tilde{E}$, then we denote by $t \mapsto x(t, \tau, \tilde{x}, v(\cdot, \cdot))$ the (maximal) trajectory of (31) such that $x(\tau, \tau, \tilde{x}, v(\cdot, \cdot)) = \tilde{x}$ as well (if this trajectory is well-defined). From conditions 2) and 3) of theorem 4.1, it follows that

$$z(t) = x(t, T, x^T, v_0(\cdot)) \quad \text{for all } t \in I, \quad (34)$$

with

$$v_0(t) := 0, \quad t \in I, \quad (35)$$

and then, using the Gronwall-Bellmann lemma, we get the existence of $\delta > 0$ such that the trajectory $t \mapsto x(t, T, x^T, v(\cdot))$ of system (32) is well-defined for all $t \in I$, and

$$\begin{aligned} (t, x(t, T, x^T, v(\cdot))) \in E \quad \text{and} \quad |x(t, T, x^T, v(\cdot)) - z(t)| &< \frac{\varepsilon_2}{2} \\ \text{for all } t \in I, \quad \text{whenever} \quad \|v(\cdot) - v_0(\cdot)\|_{L_\infty(I; \mathbf{R}^1)} &< \delta \end{aligned} \quad (36)$$

for every $v(\cdot) \in L_\infty(I; \mathbf{R}^1)$ where $\varepsilon_2 > 0$ was defined in (23).

By definition, put:

$$\mathcal{T} := \{(t, x) \in I \times \mathbf{R}^n \mid t \in I; \quad |x - z(t)| \leq \varepsilon_2\}. \quad (37)$$

Lemma 6.1. *There exist a curve $x(\cdot) \in C(I; \mathbf{R}^n)$, a finite sequence of open sets $\{\mathcal{T}_i\}_{i=1}^N$ of the form $\mathcal{T}_i =]\tau_i - \alpha_i, \tau_i + \alpha_i[\times \Omega_{\beta_i}(x_i)$ with some $(\tau_i, x_i) \in \mathcal{T}$, $i = 1, \dots, N$, a finite sequence of numbers τ_i^* in $[t_1, T]$, $i = 1, \dots, N+1$, N finite sequences of vector fields $\nu_i = \{\nu_i^k(\cdot)\}_{k=1}^{k_i}$ each of which belongs to $\Delta_{n-p-1}(\cdot)$ (i.e. $\nu_i^k(\cdot) \in \Delta_{n-p-1}(\cdot)$, $k=1, \dots, k_i$, $i=1, \dots, N$), and N finite sequences of numbers $\mu_i = \{\mu_i^k\}_{k=1}^{k_i}$, $\mu_i^k \geq 0$, $k=1, \dots, k_i$, $i=1, \dots, N$, such that*

- 1) $\overline{\Omega_{\beta_i}(x_i)} \subset \mathcal{D}_{\nu_i^1}$, $k = 1, \dots, k_i$; $\bigcup_{i=1}^N \mathcal{T}_i \subset E$, $i=1, \dots, N$
- 2) $\tau_1^* = T > \tau_2^* > \dots > \tau_N^* > \tau_{N+1}^* = t_1$; $x(T) = x^T$; and $(t, x(t)) \in \mathcal{T}_i$, whenever $t \in [\tau_{i+1}^*, \tau_i^*]$, $i=1, \dots, N$
- 3) For every $i=1, \dots, N$, and every $t \in]\tau_{i+1}^*, \tau_i^*[$, $\dot{x}(t)$ is well-defined, and

$$\frac{\partial \varphi_j}{\partial x}(t, x(t)) \dot{x}(t) = \frac{\partial \varphi_j}{\partial x}(t, x(t)) a(x(t)), \quad j=1, \dots, p-1, \quad t \in]\tau_{i+1}^*, \tau_i^*[, \quad i=1, \dots, N, \quad (38)$$

$$\frac{\partial \varphi_p}{\partial t}(t, x(t)) + \frac{\partial \varphi_p}{\partial x}(t, x(t)) \dot{x}(t) = v_i(t, x(t)), \quad t \in]\tau_{i+1}^*, \tau_i^*[, \quad i=1, \dots, N, \quad (39)$$

$$\frac{\partial \varphi_j}{\partial t}(t, x(t)) + \frac{\partial \varphi_j}{\partial x}(t, x(t)) \dot{x}(t) = 0, \quad j=p+1, \dots, n, \quad t \in]\tau_{i+1}^*, \tau_i^*[, \quad i=1, \dots, N, \quad (40)$$

where $v_i(t, x)$, $i = 1, \dots, N$ are given by

$$v_i(t, x) = \frac{\partial \varphi_p}{\partial t}(t, x) + \frac{\partial \varphi_p}{\partial x}(t, x) (\Phi_{\nu_i}^{\mu_i})_* a(\Phi_{\nu_i}^{-\mu_i}(x)) \quad \text{for all } (t, x) \in \mathcal{T}_i, \quad i=1, \dots, N \quad (41)$$

and satisfy the conditions

$$|v_i(t, x)| < \delta \quad \text{for each } (t, x) \in \mathcal{T}_i, \quad i=1, \dots, N \quad (42)$$

with δ defined in (36).

(According to the definition of the diffeomorphisms $\Phi_{\nu_i}^{\mu_i}(\cdot)$ and $\Phi_{\nu_i}^{-\mu_i}(\cdot)$ for finite sequences $\nu_i = \{\nu_i^k(\cdot)\}_{k=1}^{k_i}$, $\mu_i = \{\mu_i^k\}_{k=1}^{k_i}$ which was given in section 2, we have: $\Phi_{\nu_i}^{-\mu_i} = \Phi_{\nu_i^{k_i}}^{-\mu_i^{k_i}} \circ \dots \circ \Phi_{\nu_i^1}^{-\mu_i^1}$; $\Phi_{\nu_i}^{\mu_i} = \Phi_{\nu_i^1}^{\mu_i^1} \circ \dots \circ \Phi_{\nu_i^{k_i}}^{\mu_i^{k_i}}$)

First we assume that lemma 6.1 is already proved, and prove lemma 5.1. The proof of lemma 6.1, in turn, is based on condition (B) for system (3) and is given in Appendix (we point out that it is a modification of the proofs of lemmas 3.4, and 3.1.1 from [20]; in particular, condition (B) allows us to find $v_i(t, x)$ given by (41), and satisfying (42)). Let $x(\cdot)$ be a curve from lemma 6.1. To make the proof of lemma 5.1 clearer, we assume that $k_i = 1$, $i = 1, \dots, N$, i.e. each sequence $\mu_i = \{\mu_i^k\}_{k=1}^{k_i}$ and $\nu_i = \{\nu_i^k(\cdot)\}_{k=1}^{k_i}$ consists of one element only, and then, to simplify the notation, we put $\mu_i := \mu_i^1$, $\nu_i(\cdot) := \nu_i^1(\cdot)$ (however, we will explain how we can adjust our construction in the general case)

Put $v(t) := v_i(t, x(t))$, $t \in]\tau_{i+1}^*, \tau_i^*[$, $i = 1, \dots, N$. Then, $v(\cdot)$ is a piecewise continuous open-loop control. Combining (42), and (36), and taking into account (38)-(40), and condition 2) of lemma 6.1, we get

$$|x(t) - z(t)| < \frac{\varepsilon_2}{2} \quad \text{for all } t \in I \quad (43)$$

Let N_0 in $\{1, \dots, N\}$ be such that $\tau_{N_0+1}^* \leq t_1 + \sigma < \tau_{N_0}^*$. Without loss of generality, we assume that $\tau_{N_0+1}^* = t_1 + \sigma$; otherwise, with slight abuse of notation, we put by definition $\tau_{N_0+1}^* := t_1 + \sigma$, whereas τ_i^* , $i=1, \dots, N_0$ are the same (the terminal point of the curve $y(\cdot)$ mentioned in lemma 5.1 is $t_1 + \sigma$, and, therefore, we should deal with $[t_1 + \sigma, T]$ instead of $[t_1, T]$ in this section).

Take any $\varepsilon_3 > 0$ such that $\varepsilon_3 < \frac{\varepsilon_2}{2}$. For any $\tau \leq t$ in $[t_1 + \sigma, T]$, we put by definition:

$$\Gamma_t^\tau = \{(s, z) \in [\tau, t] \times \mathbf{R}^n \mid s \in [\tau, t]; \quad |z - x(t)| < \varepsilon_3\}; \quad \Gamma := \Gamma_{t_1 + \sigma}^T \quad (44)$$

By the construction, $(\tau_{i+1}^*, x(\tau_{i+1}^*)) \in \mathcal{T}_i \cap \mathcal{T}_{i+1}$, $i=1, \dots, N_0-1$; therefore ε_3 in $]0, \frac{\varepsilon_2}{2}[$ can be chosen such that Γ in (44) satisfies the conditions

$$\bar{\Gamma}_{\tau_1^*}^{\tau_1^*} \subset \mathcal{T}_1; \quad \bar{\Gamma}_{\tau_{i+1}^*}^{\tau_{i+1}^*} \subset \mathcal{T}_i \cap \mathcal{T}_{i+1}, \quad i = 1, \dots, N_0 - 1; \quad \bar{\Gamma}_{\tau_{N_0+1}^*}^{\tau_{N_0+1}^*} \subset \mathcal{T}_{N_0}; \quad (45)$$

$$\Gamma \subset \bigcup_{i=1}^{N_0} \mathcal{T}_i. \quad (46)$$

Put $\sigma^0 := \min\{\frac{\tau_i^* - \tau_{i+1}^*}{2}, \quad 1 \leq i \leq N_0\}$. Then, there exists $M > 0$ such that, for every sequence $\varkappa = \{\sigma_i\}_{i=1}^{N_0+1}$ satisfying the conditions $0 < \sigma_i < \sigma^0$, $i=1, \dots, N_0+1$, there are a smooth time-varying vector field $\nu_\varkappa(t, \cdot)$, $t \in [t_1 + \sigma, T]$, and a smooth function $\mu_\varkappa(\cdot) \in C^\infty([t_1 + \sigma, T]; [0, +\infty[)$ such that

$$\mu_\varkappa(T) = \mu_\varkappa(t_1 + \sigma) = 0; \quad \text{and} \quad \mu_\varkappa(t) = \mu_i,$$

$$\text{whenever} \quad t \in [\tau_{i+1}^* + \sigma_{i+1}, \tau_i^* - \sigma_i], \quad i = 1, \dots, N_0; \quad (47)$$

$$\nu_\varkappa(t, \cdot) \in \Delta_{n-p-1}(\cdot), \quad \text{whenever} \quad t \in [t_1 + \sigma, T]; \quad (48)$$

$$\nu_\varkappa(t, x) = \nu_i(x), \quad \text{whenever} \quad x \in \mathcal{D}_{\nu_i}, \quad t \in [\tau_{i+1}^* + \sigma_{i+1}, \tau_i^* - \sigma_i], \quad i = 1, \dots, N_0, \quad (49)$$

and such that the feedback control $v_\varkappa(t, x)$ given by

$$v_\varkappa(t, x) = \frac{\partial \varphi_p}{\partial t}(t, x) + \frac{\partial \varphi_p}{\partial x}(t, x) \left(\Phi_{\nu_\varkappa(t)}^{\mu_\varkappa(t)} \right)_* a(\Phi_{\nu_\varkappa(t)}^{-\mu_\varkappa(t)}(x)) \quad (50)$$

satisfies the condition

$$\max\{|F(t, x, v_\varkappa(t, x))| \mid t \in [t_1 + \sigma, T]; \quad x \in \overline{\Omega_{\varepsilon_3}(x(t))}\} \leq M, \quad (51)$$

where M is given by

$$M := \max\{|F(t, x, v)| \mid v = \frac{\partial \varphi_p}{\partial t}(t, x) + \frac{\partial \varphi_p}{\partial x}(t, x) (\Phi_{\nu_i}^\mu)_* a(\Phi_{\nu_i}^{-\mu}(x)), \quad (t, x) \in \bar{\mathcal{T}}_i, \quad 0 \leq \mu \leq \mu_i; \quad i=1, \dots, N_0\}. \quad (52)$$

For instance, given $\varkappa = \{\sigma_i\}_{i=1}^{N_0+1}$ with small enough $\sigma_i \in]0, \sigma^0[$, take any functions $\lambda_i(\cdot) \in C^\infty([t_1 + \sigma, T]; \mathbf{R})$, $i = 1, \dots, N_0$, and $\lambda(\cdot) \in C^\infty([t_1 + \sigma, T]; \mathbf{R})$ such that

$$\sum_{i=1}^{N_0} \lambda_i(t) = 1, \quad \text{and} \quad \lambda_i(t) \geq 0, \quad i = 1, \dots, N_0, \quad \text{for all } t \in [t_1 + \sigma, T];$$

$$\begin{aligned}
\lambda_i(t) &= 1, & \text{whenever } t \in [\tau_{i+1}^* + \frac{\sigma_{i+1}}{2}, \tau_i^* - \frac{\sigma_i}{2}], \quad i = 2, \dots, N_0 - 1; \quad \lambda_1(t) &= 1, \\
& \text{whenever } t \in [\tau_2^* + \frac{\sigma_2}{2}, \tau_1^*]; & \lambda_{N_0}(t) &= 1 & \text{whenever } t \in [\tau_{N_0+1}^*, \tau_{N_0}^* - \frac{\sigma_{N_0}}{2}]; \\
\lambda_i(t) + \lambda_{i+1}(t) &= 1, & \text{whenever } t \in [\tau_{i+1}^* - \frac{\sigma_{i+1}}{2}, \tau_{i+1}^* + \frac{\sigma_{i+1}}{2}], \quad i &= 1, \dots, N_0 - 1; \\
\lambda(t) &= 1 & \text{whenever } t \in \bigcup_{i=1}^{N_0} [\tau_{i+1}^* + \sigma_{i+1}, \tau_i^* - \sigma_i]; \\
0 \leq \lambda(t) \leq 1 & & \text{whenever } t \in [t_1 + \sigma, T]; \\
\lambda(t) &= 0 & \text{whenever } t \in \bigcup_{i=1}^{N_0-1} [\tau_{i+1}^* - \frac{\sigma_{i+1}}{2}, \tau_{i+1}^* + \frac{\sigma_{i+1}}{2}]; \\
\lambda(T) &= \lambda(t_1 + \sigma) = 0.
\end{aligned}$$

Then $\mu_{\varkappa}(\cdot)$ and $\nu_{\varkappa}(\cdot, \cdot)$ given by $\mu_{\varkappa}(t) = \sum_{i=1}^{N_0} \mu_i \lambda_i(t) \lambda(t)$, and $\nu_{\varkappa}(t, x) = \sum_{i=1}^{N_0} \lambda_i(t) \nu_i(x)$, $t \in [t_1 + \sigma, T]$, $x \in \mathcal{D}_{\nu_i}$ satisfy (47)-(52).

Let us remark that our vector field $\nu_{\varkappa}(t)$ is actually time-varying only around the moments of switching τ_i^* , which allows us to construct the smooth feedback control (50). If each ν_i were a sequence of vector fields $\{\nu_i^k(\cdot)\}_{k=1}^{k_i}$, we would have to take into account each switching from $\nu_i^k(\cdot)$ to $\nu_i^{k+1}(\cdot)$. The above-mentioned convex combinations would become more complicated, but the idea would be the same.

To simplify the notation, put $\nu_{\varkappa}(t) := \nu_{\varkappa}(t, \cdot)$.

Lemma 6.2 *There exists $\varkappa = \{\sigma_i\}_{i=1}^{N_0+1}$ with small enough σ_i , $0 < \sigma_i < \sigma^0$, $i = 1, \dots, N_0 + 1$ such that the corresponding trajectory $t \mapsto x_{\varkappa}(t) := x(t, T, x^T, v_{\varkappa}(\cdot, \cdot))$ of system (31) with the (smooth) feedback control $v_{\varkappa}(\cdot, \cdot)$ given by (50) and with the "initial" condition $x_{\varkappa}(T) = x^T$ is well-defined for all $t \in [t_1 + \sigma, T]$, and satisfies the condition*

$$|x_{\varkappa}(t) - x(t)| < \varepsilon_3 \quad \text{for all } t \in [t_1 + \sigma, T] \quad (53)$$

(which implies $(t, x_{\varkappa}(t)) \in \Gamma$, whenever $t \in [t_1 + \sigma, T]$).

To prove lemma 6.2, we just note that

$$x_{\varkappa}(t) = x(t, T, x^T, v_{\varkappa}(\cdot, \cdot)) \quad \text{and} \quad x(t) = x(t, T, x^T, v(\cdot, \cdot)), \quad \text{for all } t \in [t_1 + \sigma, T]$$

and $v_{\varkappa}(t, x) = v(t, x)$, whenever $t \in \bigcup_{i=1}^{N_0} [\tau_{i+1}^* + \sigma_{i+1}, \tau_i^* - \sigma_i]$, $x \in \Omega_{\varepsilon_3}(x(t))$, where

$$v(t, x) := v_i(t, x), \quad \text{whenever } \tau_{i+1}^* < t \leq \tau_i^*, \quad (t, x) \in \mathcal{T}_i, \quad i = 1, \dots, N_0, \quad (54)$$

with $v_i(t, x)$ defined in (41). In addition,

$$\max\{|F(t, x, v_{\varkappa}(t, x))| \mid t \in [t_1 + \sigma, T], \quad x \in \overline{\Omega_{\varepsilon_3}(x(t))}\} \leq M;$$

$$\max\{|F(t, x, v(t, x))| \mid t \in [t_1 + \sigma, T], \quad x \in \overline{\Omega_{\varepsilon_3}(x(t))}\} \leq M;$$

Therefore, if $\sigma_i > 0$ are small enough, then $t \mapsto x_{\varkappa}(t)$ is well-defined on $[t_1 + \sigma, T]$, and $\|x_{\varkappa}(\cdot) - x(\cdot)\|_{C(I; \mathbf{R}^n)}$ is small enough, which can be proved by the standard argument based on the Gronwall-Bellmann lemma. The proof of lemma 6.2 is complete.

Finally, we put: $\mu(t) := \mu_{\varkappa}(t)$, $\nu(t) := \nu_{\varkappa}(t)$, $t \in [t_1 + \sigma, T]$, with $\mu_{\varkappa}(t)$, $\nu_{\varkappa}(t)$ from lemma 6.2, and

$$y(t) := \Phi_{\nu(t)}^{-\mu(t)}(x_{\varkappa}(t)), \quad \text{whenever } t \in [t_1 + \sigma, T] \quad (55)$$

$$\tilde{\psi}_j(t, y) := \varphi_j(t, \Phi_{\nu(t)}^{\mu(t)}(y)), \quad j = 1, \dots, p, \quad \text{whenever } y \in \Phi_{\nu(t)}^{-\mu(t)}(\Omega_{\varepsilon_3}(x_{\varkappa}(t))) \quad (56)$$

Let us show that $y(\cdot)$ defined by (55) satisfies conditions 1), 2), 3) of lemma 5.1. Indeed, taking into account that $x_{\varkappa}(t) = \Phi_{\nu(t)}^{\mu(t)}(y(t))$, we obtain from (55), and from (31):

$$\begin{aligned} & \frac{\partial \varphi_j(t, \Phi_{\nu(t)}^{\mu(t)}(y(t)))}{\partial x} \left(\left(\frac{\partial \Phi_{\nu(t)}^{\mu(t)}}{\partial t} \right) (y(t)) + \left(\Phi_{\nu(t)}^{\mu(t)} \right)_* \dot{y}(t) \right) = \\ & = \frac{\partial \varphi_j(t, \Phi_{\nu(t)}^{\mu(t)}(y(t)))}{\partial x} a(\Phi_{\nu(t)}^{\mu(t)}(y(t))), \quad j=1, \dots, p-1, \quad t \in [t_1 + \sigma, T] \end{aligned} \quad (57)$$

By the construction, $\nu(t) \in \Delta_{n-p-1}$, for all t ; therefore

$$\frac{\partial \varphi_j(t, \Phi_{\nu(t)}^{\mu(t)}(y))}{\partial x} \left(\frac{\partial \Phi_{\nu(t)}^{\mu(t)}(y)}{\partial t} \right) = 0 \quad j=1, \dots, p \quad (58)$$

for every admissible y and t . In addition, from (A), (B) it follows that

$$a \left(\Phi_{\nu(t)}^{\mu(t)}(y(t)) \right) - \left(\Phi_{\nu(t)}^{\mu(t)} \right)_* a(y(t)) \in \Delta_{n-p}(\Phi_{\nu(t)}^{\mu(t)}(y(t)))$$

Therefore, we get from (57), (58)

$$\begin{aligned} & \frac{\partial \varphi_j(t, \Phi_{\nu(t)}^{\mu(t)}(y(t)))}{\partial x} \left(\Phi_{\nu(t)}^{\mu(t)} \right)_* \dot{y}(t) = \frac{\partial \varphi_j(t, \Phi_{\nu(t)}^{\mu(t)}(y(t)))}{\partial x} \left(\Phi_{\nu(t)}^{\mu(t)} \right)_* a(y(t)), \quad j=1, \dots, p-1, \\ & t \in [t_1 + \sigma, T]. \end{aligned}$$

Combining this with (58), (31), (50), (55) we obtain that the last equality holds for $j = p$ as well. On the other hand, by the definition of $\tilde{\psi}_j(t, y)$, the obtained equalities

$$\begin{aligned} & \frac{\partial \varphi_j(t, \Phi_{\nu(t)}^{\mu(t)}(y(t)))}{\partial x} \left(\Phi_{\nu(t)}^{\mu(t)} \right)_* \dot{y}(t) = \frac{\partial \varphi_j(t, \Phi_{\nu(t)}^{\mu(t)}(y(t)))}{\partial x} \left(\Phi_{\nu(t)}^{\mu(t)} \right)_* a(y(t)), \\ & j=1, \dots, p, \quad t \in [t_1 + \sigma, T] \end{aligned}$$

are equivalent to

$$\frac{\partial \tilde{\psi}_j}{\partial y}(t, y(t)) \dot{y}(t) = \frac{\partial \tilde{\psi}_j}{\partial y}(t, y(t)) a(y(t)), \quad j = 1, \dots, p. \quad (59)$$

Finally, by the construction, $\mu(T) = 0$, which yields $y(T) = \Phi_{\nu(T)}^{-\mu(T)}(x_{\varkappa}(T)) = x_{\varkappa}(T) = x^T$, and $\mu(t_1 + \sigma) = 0$, which yields: $y(t_1 + \sigma) = \Phi_{\nu(t_1 + \sigma)}^{-\mu(t_1 + \sigma)}(x_{\varkappa}(t_1 + \sigma)) = x_{\varkappa}(t_1 + \sigma)$.

Combining this with (43) and (53), we obtain: $y(t_1 + \sigma) \in \overline{\Omega_{\varepsilon_2}(z(t_1 + \sigma))}$, which yields condition 3) of lemma 5.1. Let $(t, x) \mapsto \psi(t, x) = (\psi_1(t, x), \dots, \psi_n(t, x)) \in \mathbf{R}^n$ be any map of class C^1 defined in some neighborhood $\tilde{G} \subset [t_1 + \sigma, T] \times \mathbf{R}^n$ of the set $\{(t, x) \in [t_1 + \sigma, T] \times$

$\mathbf{R}^n \mid x = y(t), t \in [t_1 + \sigma, T]$ such that, for every fixed $t \in I$, we have $\psi(t, y(t)) = 0$, and the map $x \mapsto (\psi_1(t, x), \dots, \psi_n(t, x))$ defines canonical coordinates for system (3) in the neighborhood $\tilde{G}_t := \{z \in \mathbf{R}^n \mid (t, z) \in \tilde{G}\}$ of $y(t)$. (Since $y(\cdot)$ is already defined, we can easily pick such a map following the same pattern as that proposed in section 4 when proving theorem 3.1 as a corollary of theorem 4.1. For this, we should consider the map $(t_1, \dots, t_n) \mapsto (\Phi_{\omega_n}^{t_n} \circ \Phi_{\omega_{n-1}}^{t_{n-1}} \circ \dots \circ \Phi_{\omega_1}^{t_1})(y(t))$ for each fixed $t \in [t_1 + \sigma, T]$, where $\omega_i(\cdot)$, $i=1, \dots, n$ are any vector fields on \mathbf{R}^n such that $\Delta_i(x) = \text{span}\{\omega_{n-i}(x), \omega_{n-i+1}(x), \dots, \omega_n(x)\}$ for all $x \in \mathbf{R}^n$, $i=0, \dots, n-1$. This map is a local diffeomorphism in some neighborhood of $t_i=0$, $i=1, \dots, n$, and the inverse map $x \mapsto \psi(t, x)$ defines canonical coordinates in a neighborhood of $y(t)$ for every fixed $t \in [t_1 + \sigma, T]$, and satisfies the condition $\psi(t, y(t)) = 0 \in \mathbf{R}^n$, $t \in [t_1 + \sigma, T]$). Then, since the definition of the Lie derivative is coordinate-free, we obtain from (59)

$$\begin{aligned} \frac{\partial \psi_j(t, y(t))}{\partial x} \dot{y}(t) &= \frac{\partial \psi_j(t, y(t))}{\partial x} a(y(t)), \quad j = 1, \dots, p, \quad t \in [t_1 + \sigma, T] \\ \frac{d\psi_j(t, y(t))}{dt} &= 0, \quad j = 1, \dots, n, \quad t \in [t_1 + \sigma, T] \end{aligned}$$

which yields conditions 1), and 2) of lemma 5.1.

Thus $y(\cdot)$, and $\psi(\cdot, \cdot) = (\psi_1(\cdot, \cdot), \dots, \psi_n(\cdot, \cdot))$ satisfy conditions 1), 2), 3) of lemma 5.1. The proof of lemma 5.1 is complete.

7. Appendix

7.1. Proof of lemma 6.1.

From conditions (A),(B) it follows that for each (t, x) in $\mathcal{T} = \{(t, x) \mid t \in I, |x - z(t)| \leq \varepsilon_2\}$ (see (37)) there exist a smooth vector field $\nu_{t,x}(\cdot) \in \Delta_{n-p-1}(\cdot)$, and a point $y = \Phi_{\nu_{t,x}}^{-\mu_{t,x}}(x)$ with some $\mu_{t,x} \geq 0$ such that

$$\frac{\partial \varphi_p}{\partial t}(t, x) + \frac{\partial \varphi_p}{\partial x}(t, x) \left(\Phi_{\nu_{t,x}}^{\mu_{t,x}} \right)_* a(\Phi_{\nu_{t,x}}^{-\mu_{t,x}}(x)) = 0.$$

(Actually, there is a finite sequence $\nu_{t,x}$ of vector fields from $\Delta_{n-p-1}(\cdot)$ and the corresponding sequence, of nonnegative numbers, which we denote by $\mu_{t,x}$, satisfying the above-mentioned equality, as in the formulation of lemma 6.1. However, to make the arguments clearer, we again assume (without loss of generality) that each of these sequences consists on one element only, then we denote these elements by $\nu_{t,x}$, and $\mu_{t,x}$ respectively. The proof for the general case is the same). Let $\mathcal{T}_{t,x}$ be an open set of the form $\mathcal{T}_{t,x} =]t - \alpha, t + \alpha[\times \Omega_\beta(x)$ with some $\alpha > 0$, $\beta > 0$ such that for its closure $\bar{\mathcal{T}}_{t,x} = [t - \alpha, t + \alpha] \times \bar{\Omega}_\beta(x)$ we get $\Phi_{\nu_{t,x}}^{-\mu}(\bar{\Omega}_\beta(x)) \subset \mathcal{D}_{\nu_{t,x}}$ for all $\mu \in [0, \mu_{t,x}]$, and

$$\left| \frac{\partial \varphi_p}{\partial t}(s, z) + \frac{\partial \varphi_p}{\partial x}(s, z) \left(\Phi_{\nu_{t,x}}^{\mu_{t,x}} \right)_* a(\Phi_{\nu_{t,x}}^{-\mu_{t,x}}(z)) \right| < \delta \quad \text{for all } (s, z) \in \bar{\mathcal{T}}_{t,x}, \quad (60)$$

where δ is defined in (36). Since $\mathcal{T} \subset \bigcup_{(t,x) \in \mathcal{T}} \mathcal{T}_{t,x}$, and \mathcal{T} is a compact set, there exists a finite

open subcovering $\{\mathcal{T}_r = \mathcal{T}_{t_r, x_r}\}_{r=1}^{r_0}$ such that $\mathcal{T} \subset \bigcup_{r=1}^{r_0} \mathcal{T}_r$. To simplify the notation, we put by

definition:

$$\mathcal{T}_r := \mathcal{T}_{t_r, x_r}, \quad \nu_r(\cdot) := \nu_{t_r, x_r}(\cdot), \quad \mu_r := \mu_{t_r, x_r}, \quad y_r := \Phi_{\nu_r}^{-\mu_r}(x_r), \quad r = 1, \dots, r_0. \quad (61)$$

Then

$$\left| \frac{\partial \varphi_p}{\partial t}(t, x) + \frac{\partial \varphi_p}{\partial x}(t, x) (\Phi_{\nu_r}^{\mu_r})_* a(\Phi_{\nu_r}^{-\mu_r}(x)) \right| < \delta \quad \text{for all } (t, x) \in \overline{\mathcal{T}}_r \quad (62)$$

By definition, put:

$$L := \frac{1}{2(M+1)}, \quad \text{where } M := \max\{ |F(t, x, v)| \mid v = \frac{\partial \varphi_p(t, x)}{\partial t} + \frac{\partial \varphi_p(t, x)}{\partial x} (\Phi_{\nu_r}^{\mu_r})_* a(\Phi_{\nu_r}^{-\mu_r}(x)), \quad (t, x) \in \overline{\mathcal{T}}_r, \quad 0 \leq \mu \leq \mu_i, \quad i=1, \dots, r_0 \} \quad (63)$$

Given any (t, x) in $\bigcup_{r=1}^{r_0} \mathcal{T}_r$, define $\theta_{t,x}(\cdot)$ and $\tau_{t,x}(\cdot)$ as follows

$$\theta_{t,x}(z) := t + \alpha_{t,x}^0 - L|z - x|; \quad \tau_{t,x}(z) := t - \alpha_{t,x}^0 + L|z - x| \quad \text{for all } z \in \mathbf{R}^n,$$

where $\alpha_{t,x}^0 > 0$ is a small positive number such that for each set

$$S_{t,x} := \{(s, z) \in I \times \mathbf{R}^n \mid \tau_{t,x}(z) < s < \theta_{t,x}(z)\}$$

there is $r=r(t, x) \in \{1, \dots, r_0\}$ such that $S_{t,x} \subset \mathcal{T}_{r(t,x)}$. Since $\mathcal{T} \subset \bigcup_{(t,x) \in \mathcal{T}} S_{t,x}$, and \mathcal{T} is compact,

there is a finite open covering $\{S_{t_m, x_m}\}_{m=1}^{m_0}$ of \mathcal{T} : $\mathcal{T} \subset \bigcup_{m=1}^{m_0} S_{t_m, x_m}$. Again, in order to simplify the notation, we put

$$\begin{aligned} \theta_m(\cdot) &= \theta_{t_m, x_m}(\cdot); \quad \tau_m := \tau_{t_m, x_m}(\cdot); \quad S_m := \{(s, z) \in I \times \mathbf{R}^n \mid \tau_m(z) \leq s \leq \theta_m(z)\}; \\ \mathcal{T}_m &:= \mathcal{T}_{r(m)}; \quad \nu_m(\cdot) := \nu_{r(m), x_m}(\cdot); \quad \mu_m := \mu_{r(m), x_m}; \quad z_m := x_{r(m), x_m} \end{aligned} \quad (64)$$

Thus, from now on, we deal with notation (64), and notation (61) is no longer valid. Let us remark that each $\theta_m(\cdot)$ and each $\tau_m(\cdot)$ satisfy the global Lipschitz condition

$$\forall y \in \mathbf{R}^n \quad \forall z \in \mathbf{R}^n \quad |\tau_j(y) - \tau_j(z)| \leq L |y - z|; \quad |\theta_j(y) - \theta_j(z)| \leq L |y - z|, \quad j=1, \dots, m_0.$$

Let Ξ be the system of all the sets given by

$$\begin{aligned} \Sigma_{\Theta(\cdot), \vartheta(\cdot), A_\Theta, A_\vartheta} &:= \{(s, z) \in \mathbf{R} \times \mathbf{R}^n \mid \vartheta(z) \leq s \leq \Theta(z)\} \setminus \{(s, z) \in \mathbf{R} \times \mathbf{R}^n \mid (s = \\ &= \vartheta(z), z \in A_\vartheta) \text{ or } (s = \Theta(z), z \in A_\Theta)\} \end{aligned} \quad (65)$$

where $\Theta(\cdot)$, and $\vartheta(\cdot)$ run through the set of all the functions of class $C(\mathbf{R}^n; I)$ such that, for all $(y, z) \in \mathbf{R}^n \times \mathbf{R}^n$,

$$|\Theta(y) - \Theta(z)| \leq L |y - z| \quad \text{and} \quad |\vartheta(y) - \vartheta(z)| \leq L |y - z|, \quad \text{for all } y \in \mathbf{R}^n, z \in \mathbf{R}^n, \quad (66)$$

(with L defined in (63)) and $A_\Theta \subset \mathbf{R}^n$, $A_\vartheta \subset \mathbf{R}^n$, run through the set of all subsets of \mathbf{R}^n .

Note that, if $\vartheta_j(\cdot)$, $j=1, \dots, N$, are some functions of \mathbf{R}^n to \mathbf{R} such that

$$\forall y \in \mathbf{R}^n \quad \forall z \in \mathbf{R}^n \quad |\vartheta_j(y) - \vartheta_j(z)| \leq L |y - z|, \quad j = 1, \dots, N,$$

then, we obtain:

$$\forall y \in \mathbf{R}^n \quad \forall z \in \mathbf{R}^n \quad \left| \max_{j=1,\dots,N} \{\vartheta_j(y)\} - \max_{j=1,\dots,N} \{\vartheta_j(z)\} \right| \leq L |y - z|,$$

and

$$\forall y \in \mathbf{R}^n \quad \forall z \in \mathbf{R}^n \quad \left| \min_{j=1,\dots,N} \{\vartheta_j(y)\} - \min_{j=1,\dots,N} \{\vartheta_j(z)\} \right| \leq L |y - z|.$$

Therefore, it is easy to prove that Ξ satisfies the following conditions: (a) $\emptyset \in \Xi$; (b) for each $\Sigma' \in \Xi$, and each $\Sigma'' \in \Xi$, we have $\Sigma' \cap \Sigma'' \in \Xi$; and (c), for every $\Sigma \in \Xi$, and every $\Sigma_1 \in \Xi$, if $\Sigma_1 \subset \Sigma$, then there exists a finite sequence $\{\Sigma_q\}_{q=2}^{q_0} \subset \Xi$ of sets from Ξ such that $\Sigma = \bigcup_{q=1}^{q_0} \Sigma_q$, and $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$, $\{i, j\} \subset \{1, \dots, q_0\}$. (In other words, Ξ is a "semiring" of sets - see [14]). Since every S_m , $m=1, \dots, m_0$ is an element of Ξ , we get the existence of a finite sequence $\{\Sigma_l\}_{l=1}^{l_0}$ of sets $\Sigma_l = \Sigma_{\Theta_l(\cdot), \vartheta_l(\cdot), A_{\Theta_l}, A_{\vartheta_l}} \in \Xi$ (see (65), (66)) such that first, $\Sigma_{l'} \cap \Sigma_{l''} = \emptyset$ for all $l' \neq l''$ in $\{1, \dots, l_0\}$; second, $\mathcal{T} \subset \bigcup_{l=1}^{l_0} \Sigma_l = \bigcup_{m=1}^{m_0} S_m$; and, third, for each l in $\{1, \dots, l_0\}$ there is $m(l)$ in $\{1, \dots, m_0\}$ such that $\Sigma_l \subset S_{m(l)}$.

For each $l \in \{1, \dots, l_0\}$, we define the feedback control $v_l(t, x)$ in $\mathcal{T}_{m(l)}$ as follows

$$v_l(t, x) = \frac{\partial \varphi_p(t, x)}{\partial t} + \frac{\partial \varphi_p(t, x)}{\partial x} \left(\Phi_{\nu_{m(l)}}^{\mu_{m(l)}} \right)_* a(\Phi_{\nu_{m(l)}}^{-\mu_{m(l)}}(x)) \quad \text{whenever } (t, x) \in \mathcal{T}_{m(l)}, \quad (67)$$

and then we define the following (discontinuous!) feedback control $v = v(t, x)$ in $\Sigma := \bigcup_{l=1}^{l_0} \Sigma_l$ for system(31)

$$v(t, x) := v_l(t, x) \quad \text{whenever } (t, x) \in \Sigma_l, \quad l = 1, \dots, l_0. \quad (68)$$

Then, the following statement holds.

Lemma 7.1 *There are a unique trajectory $x(\cdot) \in C(I; \mathbf{R}^n)$ of system (31) with the feedback law $v = v(t, x)$ given by (68), and with the initial condition $x(T) = x^T$, a unique finite sequence of indices $\{l_j\}_{j=1}^N \subset \{1, \dots, l_0\}$ and a unique sequence $T = \tau_1^* > \tau_2^* > \dots > \tau_N^* > \tau_{N+1}^* = t_1$ such that $l_i \neq l_j$ for all $i \neq j$ in $\{1, \dots, N\}$ and*

1) $\dot{x}(\cdot)$ is defined and continuous at each t in $I \setminus \{\tau_2^*, \dots, \tau_N^*\}$ and

$$(t, x(t)) \in \mathcal{T} \quad \text{and} \quad |v(t, x(t))| < \delta \quad (69)$$

for all $t \in I$.

2) For every $j=1, \dots, N$ we obtain

$$(t, x(t)) \in \Sigma_{l_j}; \quad \dot{x}(t) = F(t, x(t), v(t, x(t))) \quad \text{for all } t \in]\tau_{j+1}^*, \tau_j^*] \quad (70)$$

$$\tau_j^* = \Theta_{l_j}(z(\tau_j^*)); \quad \tau_{j+1}^* = \vartheta_{l_j}(z(\tau_{j+1}^*)); \quad (71)$$

where F is defined in (33), and $\Theta_l(\cdot)$, $\vartheta_l(\cdot)$ are given in the definition of Σ_l .

Proof of lemma 7.1. We will prove the existence and the uniqueness of $x(\cdot)$ and the corresponding $\{\Sigma_{l_i}\}_{i=1}^N$ by the induction over $i=1, \dots, N$. In addition, we will prove

(by induction on $i=1, \dots, N$) that the trajectory $x(\cdot)$ and the functions $(t, x) \mapsto s_l(t, x)$, and $(t, x) \mapsto t_l(t, x)$, given by

$$s_l(t, x) = t - \vartheta_l(x), \quad t_l(t, x) = t - \Theta_l(x), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n \quad (72)$$

satisfy the conditions

$$\begin{aligned} \frac{3(t - \tau)}{2} &\geq s_l(t, x(t)) - s_l(\tau, x(\tau)) \geq \frac{t - \tau}{2}; \quad \frac{3(t - \tau)}{2} \geq t_l(t, x(t)) - t_l(\tau, x(\tau)) \geq \\ &\geq \frac{t - \tau}{2} \quad \text{for all } t > \tau, \quad \{t, \tau\} \subset [\tau_i^*, T], \quad l = 1, \dots, l_0. \end{aligned} \quad (73)$$

For $i = 1$, we put by definition: $\tau_i^* = \tau_1^* = T$. At this stage, we have empty set of l_j , Σ_{l_j} , and empty set of equalities (70), (71), (73), $j=1, \dots, i-1$, and the algorithm for getting l_1 , Σ_{l_1} , and τ_2^* is the same as in the general case $i \geq 1$, which we consider now.

Assume that, for some $i \geq 1$, there are a unique sequence $\{l_j\}_{j=1}^{i-1}$ (for $i = 1$, the sequence is empty - see the previous paragraph) such that $l_j \neq l_q$ for all $j \neq q$ in $\{1, \dots, i-1\}$, a unique sequence $T = \tau_1^* > \tau_2^* > \dots > \tau_i^* \geq t_1$, and a trajectory $x(\cdot) \in C([\tau_i^*, T]; \mathbf{R}^n)$ such that (69), and (73) hold for all $t \in [\tau_i^*, T]$; and (70), (71) hold for all $j=1, \dots, i-1$; $\dot{x}(\cdot)$ being well-defined and continuous on $[\tau_i^*, T] \setminus \{\tau_2^*, \dots, \tau_i^*\}$ (again, for $i = 1$, we deal with empty set of equalities (70), (71), (73), and (69) becomes trivial).

If $\tau_i^* = t_1$, the proof is complete, therefore we assume that $\tau_i^* > t_1$. By the induction hypothesis, $(\tau_i^*, x(\tau_i^*)) \in \mathcal{T}$; then, since $\mathcal{T} \subset \text{int} \left(\bigcup_{l=1}^{l_0} \Sigma_l \right)$ by the construction, we get the existence of $\alpha_0 > 0$ such that $(\tau_i^* - s, x(\tau_i^*)) \in \bigcup_{l=1}^{l_0} \Sigma_l$ for all $s \in]0, \alpha_0]$. Since $\Sigma_{l'} \cap \Sigma_{l''} = \emptyset$ for all $l' \neq l''$, there are a unique $l_i \in \{1, \dots, l_0\}$ and a unique $\bar{\tau} \in [t_1, \tau_i^*[$ such that $]\bar{\tau}, \tau_i^*[\times\{x(\tau_i^*)\} \subset \Sigma_{l_i}$, and $\bar{\tau} = \vartheta_{l_i}(x(\tau_i^*))$, $\tau_i^* = \Theta_{l_i}(x(\tau_i^*))$. Since $v_{l_i}(\cdot, \cdot)$ is well-defined in $\mathcal{T}_{m(l_i)}$ (see (67)), and $(\tau_i^*, x(\tau_i^*)) \in \Sigma_{l_i} \subset \mathcal{T}_{m(l_i)}$, the trajectory $t \mapsto x(t, \tau_i^*, x(\tau_i^*), v_{l_i}(\cdot, \cdot))$ is well-defined on some maximal $]\bar{s}, \tau_i^*]$ such that $x(t, \tau_i^*, x(\tau_i^*), v_{l_i}(\cdot, \cdot)) \in \mathcal{T}_{m(l_i)}$ for all $t \in]\bar{s}, \tau_i^*]$. To simplify the notation, we put $x_{l_i}(t) := x(t, \tau_i^*, x(\tau_i^*), v_{l_i}(\cdot, \cdot))$ for $t \in]\bar{s}, \tau_i^*]$. Let us remark that from (33), (63), (66) (and from the inclusions $\Sigma_l \in \Xi$, $l=1, \dots, l_0$) it follows that, for every $l' \in \{1, \dots, l_0\}$ such that $(\tau_i^*, x(\tau_i^*)) \in \mathcal{T}_{m(l')}$ the trajectory $t \mapsto x_{l'}(t) := x(t, \tau_i^*, x(\tau_i^*), v_{l'}(\cdot, \cdot))$ satisfies the conditions

$$\begin{aligned} \frac{3(t - \tau)}{2} &\geq s_l(t, x_{l'}(t)) - s_l(\tau, x_{l'}(\tau)) \geq \frac{(t - \tau)}{2} \\ \frac{3(t - \tau)}{2} &\geq t_l(t, x_{l'}(t)) - t_l(\tau, x_{l'}(\tau)) \geq \frac{(t - \tau)}{2}, \quad t > \tau, \quad l = 1, \dots, l_0 \end{aligned} \quad (74)$$

In particular, (74) holds for $l' = l_i$ and for all t, τ in $]\bar{s}, \tau_i^*]$, which implies that $t \mapsto s_l(t, x_{l_i}(t))$, and $t \mapsto t_l(t, x_{l_i}(t))$ are strictly increasing on $]\bar{s}, \tau_i^*]$. From the induction hypothesis (see (71) for $j = 1, \dots, i-1$) it follows that $t_l(\tau_i^*, x(\tau_i^*)) = 0$, $s_l(\tau_i^*, x(\tau_i^*)) > 0$. Since $s_l(t, x) > 0$, and $t_l(t, x) < 0$ are equivalent to $t > \vartheta_l(x)$ and $t < \Theta_l(x)$ respectively, we obtain that $(t, x(t, \tau_i^*, x(\tau_i^*), v_{l_i}(\cdot, \cdot)))$ belongs to Σ_{l_i} for all $t \in]\tau_i^* - \alpha_0, \tau_i^*]$ with some $\alpha_0 > 0$, and moreover there is a unique $\tau_{i+1}^* \in]\bar{s}, \tau_i^*]$ such that $t_{l_i}(t, x_{l_i}(t)) < 0$, $s_{l_i}(t, x_{l_i}(t)) > 0$ for all $t \in]\tau_{i+1}^*, \tau_i^*]$, and $t_{l_i}(t, x_{l_i}(t)) < 0$, $s_{l_i}(t, x_{l_i}(t)) < 0$ for all $t \in]\bar{s}, \tau_{i+1}^*]$, which implies that

$$(t, x_{l_i}(t)) \in \Sigma_{l_i}, \quad \dot{x}_{l_i}(t) = F(t, x_{l_i}(t), v_{l_i}(t, x_{l_i}(t))) \quad \text{for all } t \in]\tau_{i+1}^*, \tau_i^*] \quad (75)$$

$$(t, x_{l_i}(t)) \notin \Sigma_{l_i} \quad \text{for all } t \in]\bar{s}, \tau_{i+1}^*[\quad (76)$$

Taking into account that $x_{l_i}(\tau_i^*) = x(\tau_i^*)$ by the construction, we obtain from (75) and from the induction hypothesis that our l_i , Σ_{l_i} , τ_{i+1}^* , and $x(t)$ given by $x(t) := x_{l_i}(t)$ for all $t \in [\tau_{i+1}^*, \tau_i^*]$ satisfy (69)-(71), and (73) for all $j = 1, \dots, i-1, i$.

The uniqueness of l_i , Σ_{l_i} , τ_{i+1}^* , and $x(\cdot)$ on $[\tau_{i+1}^*, \tau_i^*]$ such that (75) holds with some τ_{i+1}^* ($\tau_{i+1}^* < \tau_i^*$) follows from (74), which is true for each l' in $\{1, \dots, l_0\}$ such that $(\tau_i^*, x(\tau_i^*)) \in \mathcal{T}_{m(l')}$. Indeed, if there are $l' \neq l_i$ in $\{1, \dots, l_0\}$, $\bar{\tau}_{i+1}^* < \tau_i^*$, and $\bar{x}(\cdot) \in C([\bar{\tau}_{i+1}^*, \tau_i^*]; \mathbf{R}^n) \cap C^1([\bar{\tau}_{i+1}^*, \tau_i^*]; \mathbf{R}^n)$ such that

$$\begin{aligned} \bar{x}(\tau_i^*) &= x(\tau_i^*); \quad \bar{x}(t) \in \Sigma_{l'} \quad \text{for all } t \in]\bar{\tau}_{i+1}^*, \tau_i^*[\\ \dot{\bar{x}}(t) &= F(t, \bar{x}(t), v(t, \bar{x}(t))) = F(t, \bar{x}(t), v_{l'}(t, \bar{x}(t))), \quad t \in]\bar{\tau}_{i+1}^*, \tau_i^*[\\ \vartheta_{l'}(\bar{\tau}_{i+1}^*, \bar{x}(\bar{\tau}_{i+1}^*)) &= \bar{\tau}_{i+1}^*, \end{aligned} \quad (77)$$

then we get from (74) and from the definition of $l_i : s_{l_i}(t, \bar{x}(t)) > 0$, $t_{l_i}(t, \bar{x}(t)) < 0$ for $t \in]\bar{\tau}, \tau_i^*[$ with some $\bar{\tau} \in]\bar{\tau}_{i+1}^*, \tau_i^*[$, which yields: $(t, \bar{x}(t)) \in \Sigma_{l'} \cap \Sigma_{l_i}$ for all $t \in]\bar{\tau}, \tau_i^*[$. Since $\Sigma_{l'} \cap \Sigma_{l_i} = \emptyset$ for each $l' \neq l_i$, we obtain that $l' = l_i$, which proves the uniqueness of l_i and of Σ_{l_i} . This, in turn, implies that $\bar{x}(t) = x(t)$ for all $t \in [\hat{\tau}, \tau_i^*]$, where $\hat{\tau} := \max\{\bar{\tau}_{i+1}^*, \tau_{i+1}^*\}$. The function $s_{l_i}(t) = t - \vartheta_{l_i}(t, x(t))$ is strictly increasing on $[\bar{\tau}_{i+1}^*, \tau_i^*] \cup [\tau_{i+1}^*, \tau_i^*]$; therefore, from the equalities $\vartheta_{l_i}(\bar{\tau}_{i+1}^*, \bar{x}(\bar{\tau}_{i+1}^*)) = \bar{\tau}_{i+1}^*$, $\vartheta_{l_i}(\tau_{i+1}^*, x(\tau_{i+1}^*)) = \tau_{i+1}^*$, it follows that $\hat{\tau} = \bar{\tau}_{i+1}^* = \tau_{i+1}^*$, which proves the uniqueness of τ_{i+1}^* , and the uniqueness of $x(\cdot)$ on $[\tau_{i+1}^*, \tau_i^*]$. Let us remark that $(t, x(t)) \in \mathcal{T}$ for all $t \in [\tau_{i+1}^*, \tau_i^*]$, which follows from the inclusion $(t, x(t)) \in \mathcal{T}$ for $t \in [\tau_i^*, T]$ (see the induction assumption) and from the choice of δ (see (36), (60)). Finally, (73), and (74) with $l' = l$ yield

$$\begin{aligned} \frac{3(t-\tau)}{2} &\geq s_l(t, x(t)) - s_l(\tau, x(\tau)) \geq \frac{(t-\tau)}{2}; \quad \frac{3(t-\tau)}{2} \geq t_l(t, x(t)) - t_l(\tau, x(\tau)) \geq \\ &\geq \frac{(t-\tau)}{2} \quad \text{for all } t > \tau, \quad \{t, \tau\} \subset [\tau_{i+1}^*, T], \quad l = 1, \dots, l_0, \end{aligned}$$

which implies that $s_{l_i}(t, x(t)) > 0 > s_{l_j}(t, x(t))$, $j=1, \dots, i-1$, and therefore $l_i \neq l_j$ for all $j=1, \dots, i-1$; hence $l_j \neq l_q$ for all $q \neq j$, $\{q, j\} \subset \{1, \dots, i\}$. The construction of Σ_{l_j} , τ_{j+1}^* , and $x(t)$, $t \in [\tau_{j+1}^*, \tau_j^*]$ is complete.

Thus, we obtain by the induction sequences $\{l_j\}$, $\{\Sigma_j\}$, τ_j^* , and the trajectory $x(\cdot)$ of system (31) with the feedback control $v = v(t, x)$ given by (68) such that $x(T) = x^T$, $\tau_{j+1}^* < \tau_j^*$; $l_i \neq l_j$ for all $i \neq j$, and (69)-(71) hold for all $j = 1, 2, \dots$. If $\tau_{i+1}^* = t_1$ for some $i \in \mathbf{N}$, we put $N := i$, and the proof of lemma 6.1 is complete. Otherwise, sequences $\{\tau_i^*\}$, and $\{l_i\}$ are infinite, which is impossible because $\{l_i\} \subset \{1, \dots, l_0\}$ and $l_i \neq l_j$, whenever $i \neq j$. The proof of lemma 7.1. is complete. Finally, we put $\mathcal{T}_i := \mathcal{T}_{m(l_i)}$, and $v_i(t, x) := v_{l_i}(t, x)$, for all $(t, x) \in \mathcal{T}_i := \mathcal{T}_{m(l_i)}$, $i = 1, \dots, N$. Then we obtain from lemma 7.1 that our sequences of \mathcal{T}_i and $v_i(\cdot, \cdot)$ satisfy all the conditions of lemma 6.1. The proof of lemma 6.1. is complete.

References

- [1] Athanasov ., Willems P.I. O nelineinom upravlenii girostatom na orbite // Izvestiya RAN. Ser. Thechnicheskaya kibernetika. – 1993. – 1. – P. 16 – 23.
- [2] Borisov V.F., Zelikin M.I. Chattering arcs in the time-optimal robots control problem // Prikladnaya Matematika i Mehanika. – 1988. – V. 52, 6. – P. 939 – 946.
- [3] Brunovsky P. A classification of linear controllable systems // Kybernetika. – 1970. – Vol. 6, 3. – P. 173 – 188.
- [4] Celikovsky S., Nijmeijer H. Equivalence of nonlinear systems to triangular form: the singular case // Systems and Control Letters. – 1996. – Vol. 27. – P. 135 – 144.
- [5] Cheng D., Isidori A., Respondek W., Tarn T.J. Exact linearization of nonlinear systems with outputs // Math. Syst. Theory. – 1988. – Vol. 21, 2. – P. 63 – 83.
- [6] D’Andrea B., Levine J. C.A.D. for nonlinear systems decoupling, perturbations rejection and feedback linearization with applications to the dynamic control of a robot arm// Algebraic and Geom. Meth. Nonlinear Control Theory. M. Fliess and M. Hazewinkel eds. (Reidel, Dordrecht). – 1986. – P. 545 – 572.
- [7] Freeman R.A., Kokotovic P.V. Backstepping design of robust controllers for a class of nonlinear systems // Nonlinear Control Systems Des. Symp. (NOLCOS 92). – Bordeaux: Proc. IFAC. – 1992. – P. 307 – 312.
- [8] Fliess M., Levine J., Martin Ph., Rouchon P. Flatness and defect of nonlinear systems: introductory theory and examples. Int. J. Control, – 1995. – Vol. 61. – 6. – P. 1327 – 1361.
- [9] Gardner R.B., Shadwick W.F. An algorithm for feedback linearization // Differ. Geom. and Appl. – 1991. – Vol. 1, 2. – P. 153 – 158.
- [10] Gardner R.B., Shadwick W.F. The GS algorithm for exact linearization to Brunovsky normal form // IEEE Trans. Autom. Contr. – 1992. – Vol. 37, 2. – P. 224 – 230.
- [11] Hunt L.R., Su R., Meyer G., Global transformations of nonlinear sysems // IEEE Trans. Autom. Contr. – 1983. – V. 28, 1.
- [12] Jakubczyk B., Respondek W. On linearization of control systems // Bull. Acad. Sci. Polonaise Ser. Sci. Math. – 1980. – Vol. 28. – P. 517-522.
- [13] Kokotovic P.V., Sussmann H.J. A positive real condition for global stabilization of nonlinear systems // Syst. and Contr. Lett. – 1989. – Vol. 13, 2. – P. 125 – 133.
- [14] Kolmogorov A.N., Fomin S.V. Elements of Theory of Functions and Functional Analysis. (In Russian) – Moscow, Nauka, 1989. – 624 P.
- [15] Korobov V.I. Controllability and stability of some nonlinear systems // Differencial’nie uravnenija. – 1973. – Vol. 9, N 4. – P. 614 – 619.
- [16] Korobov V.I. Reduction of the controllability problem to a boundary problem // Differencial’nie uravnenija. – 1976. – Vol. 12, N 7. – P. 1310 – 1312.

- [17] Korobov V. I., Pavlichkov S. S. Controllability of the triangular systems that are not equivalent to the canonical systems // Vestnik Kharkovskogo Natsional'nogo Universiteta. Ser. Matematika, Prikladnaya Matematika i Mechanika. – 2000. – N 475. – P. 323-329.
- [18] Korobov V. I., Pavlichkov S. S., Schmidt W. H. The controllability problem for certain nonlinear integro-differential Volterra systems // Optimization. – 2001. – Vol. 50, N 3-4. – P. 155-186.
- [19] Korobov V. I., Pavlichkov S. S., Schmidt W. H. Global robust controllability of the triangular integro-differential Volterra systems // J. of Math. Anal. Appl. 309 (2005) 743-760.
- [20] Korobov V. I., Pavlichkov S.S. The solution of the global controllability problem for the triangular systems in the singular case. // 29 pages, Math Preprints Server Nov. 29, 2003, and ArXiv math.OC/0509064
- [21] Kovalev A.M., Scherbak V.F. Upravlyaemost', nabludaemost', identifiatsionnost' dinamicheskikh sistem. (in Russian) – Kiev: Naukova dumka, 1993.
- [22] Krishchenko A.P., Klinkovsky M.G. Preobrazovanie afinnykh sistem s upravleniem i zadacha stabilizatsii // Differentsial'nye uravneniya. – 1992. – 11. – P. 1945 – 1952.
- [23] Krishchenko A.P. Kanonicheskie vidy afinnykh sistem i zadacha stabilizatsii // International Conf. Dedicated to the 90-th Anniversary of L.S. Pontryagin. Abstracts. Optimal control and Appendices. – Moscow. – 1998. – P. 238 – 240.
- [24] Lee A.B., Marcus L. Foundations of the optimal control theory: (In Russian) – Moscow.: Nauka, 1972. – 576 p.
- [25] Lin J.-S., Kanellakopoulos I. Nonlinearities enhance parameter convergence in strict-feedback systems // IEEE Trans. Autom. Contr. – 1998. – Vol. 43, 9. – P. 1-5.
- [26] Marino R. Feedback linearization techniques in robotics and power systems // Algebraic and Geom. Meth. in Nonlinear Contr. Theory. M. Fliess and M. Hazewinkel eds. (Reidel, Dordrecht). – 1986. – P. 523 – 543.
- [27] Murray R.M. Trajectory generation for a towed cable flight control system. In Proc. IFAC World Congress, San Francisco, 1996. – P. 395-400.
- [28] Nam K., Arapostathis A. A model reference adaptive control scheme for pure-feedback nonlinear systems // IEEE Trans. Autom. Contr. – 1988. – Vol. 33, 9. – P. 803 – 811.
- [29] Nelineijnij analiz povedeniya mekhanicheskikh sistem / G.V. Gorr, A.A. Ilyuhin, A.M. Kovalyov, A.Ya. Savchenko. (In Russian) – Kiev: Naukova Dumka, 1984.
- [30] Nijmeijer H. Tracking control of mobile robots // Math. Forschungsinst. – Oberwolfach. – 1996. – 44. – P. 11 – 12.
- [31] Respondek W. Global aspects of linearization, equivalence to polynomial forms and decomposition of nonlinear control systems // Algebraic and Geom. Meth. in Nonlinear Control Theory. M. Fliess and M. Hazewinkel eds. (Reidel, Dordrecht). – 1986. – P. 257 – 284.
- [32] Singh S.N., Bossart T.C. Exact feedback linearization and control of space station using CMG // IEEE Trans. Autom. Contr. – 1993. – Vol. 38, 1. – P. 184 – 187.

- [33] Saberi A., Kokotovic P.V., Sussmann H.J. Global stabilization of partially linear composite systems // SIAM J. Contr. Optimiz. – 1990. – Vol. 28, 6. – P. 1491 – 1503.
- [34] Sternberg S. Lectures on differential geometry. (In Russian) – Moscow: Mir, 1970. – 412 P.
- [35] Tsiniias J. A theorem on global stabilization of nonlinear systems by linear feedback // Syst. and Contr. Lett. – 1991. – Vol. 17, 5. – P. 357 – 362.
- [36] Zhevnin A.A., Krishchenko A.P. Upravlyaemost' nelineinikh sistem i sintez algoritmov upravlenija // Doklady Akademii Nauk SSSR. Ser. Kibernetika i teoriya regulirovanija. – 1981. – V. 258, 4. – P. 805 – 809.